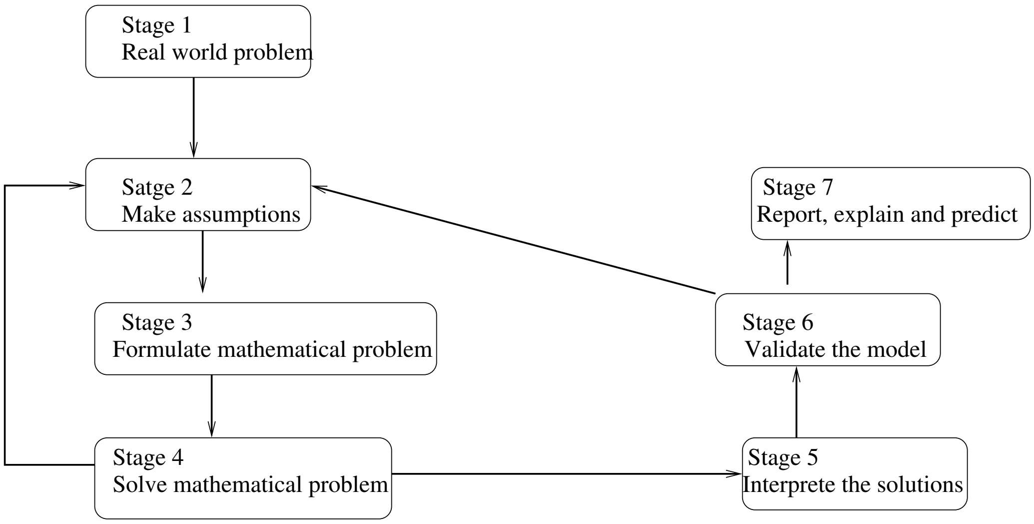
Introduction to Mathematical Modelling and Examples

Typically, a mathematical modelling process has the stages shown in the diagram below:



In Stage 4, we find the solutions. If some of the solutions are not feasible, then we go back to Stage 2 and modify the assumptions. In stage 6, we obtain the feasible solutions, but these do not match the real data. In this case, we also go back to Stage 2.

A mathematical model must take into account the important factors affecting the problem. However, it must not be made too complicated by including terms whose effects are negligible. For example, if a weather prediction model for tomorrow takes three days to solve in a supercomputer, then the model does not serve any purpose.

Example (Mathematical modelling process): Consider a ball of mass m kg falling from rest from a height of 300 m above the ground. We want to model the motion and predict the distance $x(t)$ covered in time t.

Let $x(t)$ be the distance covered measured from the top. We observe that the velocity of the ball increases as the distance x increases. Thus, as a first attempt, we postulate that $dx/dt \propto x$ with $x(0) = 0$. Thus, the model becomes

$$\frac{dx}{dt} = \alpha x,$$
$$x(0) = 0.$$

This is a well-posed problem. Due to the existence and uniqueness theorem, the solution to the problem is $x(t) \equiv 0$! However, this is an absurd result. Hence, something is not correct.

As a second attempt, we use the idea (from physics) that a freely falling body moves under the action of gravity. Then we modify the governing equation as

$$\frac{d^2x}{dt^2} = g,$$
$$x(0) = 0,$$
$$x'(0) = 0.$$

Here, g is the acceleration due to gravity. Solving, we find that $x(t) = \frac{1}{2}gt^2$. This matches the observed motion when the ball is heavy. However, when the ball is lighter, the results do not agree with the observation.

To further improve the model, we need to account for air resistance in the motion. This will be our third attempt, and we will discuss it later. This example shows the stages of a mathematical model.

Radioactive decay: The assumptions of the model are

(i) The amount of the material is so large that random fluctuations can be neglected

(ii) Process is continuous in time

(iii) Fixed rate of decay

(iv) System is isolated

Let $m(t)$ be the mass of the element at time t and $m(t_0) = m_0$ is the initial mass at time $t = t_0$. The decay is proportional to the amount present. Hence, if $k > 0$ is the proportionality constant, then

$$\frac{dm}{dt} = -km, \quad m(t_0) = m_0 > 0.$$

The negative sign on the RHS is due to the decreasing mass with time. Since the governing system is autonomous (i.e., the RHS is not explicitly a function of time t), the initial time can be taken to be $t = 0$. The above equation is separable, and the solution is

$$m(t) = m_0 \mathrm{e}^{-k(t-t_0)}, \quad t \geq t_0.$$

It is seen that $m(t) > 0$, $m'(t) = -km < 0$, and $m(t) \to 0$ as $t \to \infty$. Hence, $m(t)$ is monotonic decreasing with no turning point.

The half-life τ of the radioactive element is defined to be the time it takes to decrease by half. Thus

$$\frac{m(t+\tau)}{m(t)} = \frac{1}{2} \implies \frac{\mathrm{e}^{t+\tau-t_0}}{\mathrm{e}^{t-t_0}} = \frac{1}{2} \implies \mathrm{e}^{-k\tau} = \frac{1}{2} \implies \tau = \frac{\ln 2}{k}$$

Radiocarbon dating: All living trees absorb CO_2 from air for photosynthesis. CO_2 contains a radioactive form of carbon ^{14}C as well as common carbon ^{12}C. The radioactive ^{14}C is produced by the collision of cosmic rays with nitrogen in the atmosphere, and it decays back to nitrogen again. The half-life of ^{14}C is approximately 5700 years.

Radiocarbon dating depends on the fact that for any living tree, the ratio of ^{14}C to ^{12}C is the same as the surroundings. But when a tree dies, CO_2 from the air is no longer absorbed, although ^{14}C within the dead tree continues to decay. By calculating the amount of ^{14}C in the dead tree now, we can figure out how long ago it died. To know the amount of ^{14}C when it died, we assume that the proportion of ^{14}C in any living tree is constant. This implies that today's reading would be the same as that many years ago (not strictly true!).