Discrete modelling: Solution of linear difference equations

Let us consider a nonhomogeneous linear first-order difference equation

$$x_{n+1} = ax_n + b_n,$$

where b_n may depend on n. Suppose that a particular solution p_n of the above problem is known. Let y_n be any other solution. Then

$$y_{n+1} = ay_n + b_n \quad \text{and} \quad p_{n+1} = ap_n + b_n$$

Introducing $z_n = y_n - p_n$, we find

$$z_{n+1} = az_n$$

This gives

$$z_n = a^n z_0 = a^n(y_0 - p_0) = ca^n \implies y_n = ca^n + p_n,$$

where $c = y_0 - p_0$ is an arbitrary constant. We call y_n the general solution of the nonhomogeneous linear first-order difference equation. We see that the general solution consists of two terms. The first term, ca^n, is a solution of the homogeneous part, $x_{n+1} = ax_n$, and the second part, p_n, is a particular solution of the nonhomogeneous equation. This is true for nonhomogeneous linear difference equations of any order.

Example: Consider

$$x_{n+1} = 3x_n + 2n + 1$$

We can verify that $p_n = -(n + 1)$ is a particular solution since

$$\text{LHS} = -(n + 2) \quad \text{and} \quad \text{RHS} = -3(n + 1) + 2n + 1 = -(n + 2)$$

Hence, the general solution is $x_n = c3^n - (n + 1)$.

It is obvious that to find the solution of the nonhomogeneous equation, we need to find the particular solution. For a general b_n, this is not an easy task. However, there is a method that applies always if b_n is a combination of powers of n (like $n^0 = 1, n, n^2$,etc.), trigonometric functions of n, and powers b^n. This method is called the method of undetermined coefficients (similar to that used in ODE). Consider

$$x_{n+1} = ax_n + b_n.$$

If

$$b_n = (c_0 + c_1 n + c_2 n^2 + \cdots + C_M n^M)b^n, \qquad c_M \neq 0,$$

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then choose

$$
\begin{aligned}
p_n &= (A_0 + A_1 n + A_2 n^2 + \cdots + A_M n^M) b^n, \quad \text{for } b \neq a \\
p_n &= n(A_0 + A_1 n + A_2 n^2 + \cdots + A_M n^M) b^n, \quad \text{for } b = a.
\end{aligned}
$$

If

$$
b_n = (c_0 + c_1 n + c_2 n^2 + \cdots + C_M n^M) b^n \cos(kn) + (d_0 + d_1 n + d_2 n^2 + \cdots + d_M n^M) b^n \sin(kn),
$$

where at least one of c_M or d_M is nonzero, then choose

$$
p_n = (A_0 + A_1 n + A_2 n^2 + \cdots + A_M n^M) b^n \cos(kn) + (B_0 + B_1 n + B_2 n^2 + \cdots + b_M n^M) b^n \sin(kn).
$$

Example: Consider the example $x_{n+1} = 3x_n + 2n + 1$ where we have verified earlier that $p_n = -(n+1)$. Since $1 + 2n = (1+2n)1^n$, here we have $b = 1 \neq 3 = a$. Hence, we take $p_n = (A + Bn)$. Substituting into the equation, we find

$$
A + B(n+1) = 3(A + Bn) + 2n + 1 \implies (2A - B + 1) + (2B + 2)n = 0.
$$

The above holds for all n provided $2B + 2 = 0$, $2A - B + 1 = 0 \implies B = -1, A = -1$ and hence $p_n = -(n+1)$, the same we found earlier.

Example: To find the particular solution of $x_{n+1} = 2x_n + \sin 2n$, we taken $p_n = A\cos 2n + B\sin 2n$. Substituting, we find

$$
A\cos(2n+2) + B\sin(2n+2) = 2(A\cos 2n + B\sin 2n) + \sin 2n
$$

Expanding the terms in the LHS & rearranging, we find

$$
(A\cos 2 + B\sin 2 - 2A)\cos 2n + (-A\sin 2 + B\cos 2 - 2B - 1)\sin 2n = 0.
$$

This equation holds for all n if the terms in the parentheses vanish. Setting these terms equal to zero, we find the following system of equations:

$$
\begin{aligned}
(\cos 2 - 2)A + (\sin 2)B &= 0, \\
-(\sin 2)A + (\cos 2 - 2)B &= 1.
\end{aligned}
$$

Solutions for A and B provide the particular solution.

Example: To find the solution of $x_{n+1} = 2x_n + n2^n,\ x_0 = 1$, we first find the general solution. Since here $b = 2 = a$, we assume $p_n = n(A + Bn)2^n$. Substituting, we find

$$
\begin{aligned}
(n+1)[A + B(n+1)]2^{n+1} &= 2n(A + Bn)2^n + n2^n \\
\implies 2(n+1)[A + B(n+1)] &= 2n(A + Bn) + n \\
\implies (4B - 1)n + 2(A + B) = 0 &\implies -A = B = \frac{1}{4}
\end{aligned}
$$

Hence, $p_n = \frac{1}{4}n(n-1)2^n$ and the general solution is

$$x_n = c2^n + \frac{1}{4}n(n-1)2^n$$

Using $x_0 = 1$, we find $c = 1$ and hence the solution of the given problem is

$$x_n = 2^n + \frac{1}{4}n(n-1)2^n$$

<u>Linear second order equations</u>: Here we concentrate on homogeneous part only. The reason is that we can find a particular solution using the method of undetermined coefficients when the non-homogeneous part is of a particular form. Let the difference equation be of the form

$$x_{n+2} + \alpha x_{n+1} + \beta x_n = 0,$$

where α and β are real numbers. Assuming a solution of the form $x_n = \lambda^n$, we find the auxiliary equation

$$\lambda^2 + \alpha\lambda + \beta = 0$$

The roots are

$$\lambda_{\pm} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}$$

There are three cases to consider:

i. $\alpha^2 > 4\beta$: The real roots are distinct and the solution is

$$x_n = c_1\big(\lambda_+\big)^n + c_2\big(\lambda_-\big)^n,$$

where c_1 and c_2 are two arbitrary parameters. Two arbitrary parameters arise from a second-order difference equation. Note that $x_n \to 0$ for $n \to \infty$ when $|\lambda_{\pm}| < 1$. If $|\lambda_+| > 1$ or $|\lambda_-| > 1$, then $|x_n| \to \infty$ as $n \to \infty$.

Example: Consider $x_n = x_{n-1} + x_{n-2}, \quad x_1 = x_2 = 1, \quad n \geq 3$. This appears in Fibonacci's rabbit problem. The auxiliary equation is $\lambda^2 - \lambda - 1 = 0$ whose roots are

$$\lambda_+ = \frac{1+\sqrt{5}}{2}, \ \lambda_- = \frac{1-\sqrt{5}}{2}$$

Hence, the general solution is

$$x_n = c_1(\lambda_+)^n + c_2(\lambda_-)^n$$

Using the initial conditions, we find

$$c_1\lambda_+ + c_2\lambda_- = 1, \quad c_1\lambda_+^2 + c_2\lambda_-^2 = 1$$

Since $\lambda_+^2 = \lambda_+ + 1$ and $\lambda_-^2 = \lambda_- + 1$, the last equation gives

$$c_1\lambda^+ + c_2\lambda_- + c_1 + c_2 = 1 \implies c_1 + c_2 = 0$$

Solving with the first equation, we get

$$c_1 = \frac{1}{\lambda_+ - \lambda_-} = \frac{1}{\sqrt{5}}, \quad c_2 = -\frac{1}{\sqrt{5}}$$

Hence,

$$x_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n, \qquad n \geq 3.$$

ii. $\alpha^2 = 4\beta$: In this case we have a real double root $\lambda_+ = \lambda_- = -\frac{\alpha}{2}$. Using this, we have one solution $x_{n1} = (-\alpha/2)^n$. We can directly verify that $x_{n2} = n(-\alpha/2)^n$ is also a solution, which is independent of the previous solution. Hence, the general solution is

$$x_n = \left(-\frac{\alpha}{2}\right)^n (c_1 + c_2 n)$$

Example: For $x_{n+1} - 4x_n + 4x_{n-1} = 0$, the auxiliary equation is $\lambda^2 - 4\lambda + 4 = 0$. Solving we find $\lambda_+ = \lambda_- = 2$ and the general solution is $x_n = (c_1 + c_2 n)2^n$.

iii. $\alpha^2 < 4\beta$: Here, β must be greater than zero. The roots are complex conjugate

$$\lambda_\pm = \frac{-\alpha}{2} \pm i\frac{\sqrt{4\beta - \alpha^2}}{2}.$$

Note that $U_n^+ = (\lambda_+)^n$ and $U_n^- = (\lambda_-)^n$ are two linearly independent solutions. However, they are complex-valued. To get real-valued solutions, we take

$$x_n^+ = \frac{U_n^+ + U_n^-}{2}, \quad x_n^- = \frac{U_n^+ - U_n^-}{2i},$$

which are real-valued. Since the difference equation is linear, the linear combination of solutions is also a solution. Hence, x_n^+ and x_n^- are solutions. It can be proved easily that they are also linearly independent.

Now we write

$$\lambda_+ = \frac{-\alpha}{2} + i\frac{\sqrt{4\beta - \alpha^2}}{2} = re^{i\theta}, \quad \text{where } r = \sqrt{\beta},\ r\cos\theta = -\frac{\alpha}{2},\ r\sin\theta = \frac{\sqrt{4\beta - \alpha^2}}{2}.$$

Hence,

$$x_n^+ = \frac{U_n^+ + U_n^-}{2} = r^n \cos n\theta,\ x_n^- = \frac{U_n^+ - U_n^-}{2i} = r^n \sin n\theta.$$

Finally, a real-valued general solution is given by

$$x_n = r^n(c_1 \cos n\theta + c_2 \sin n\theta)$$

The form of the solution implies that $x_n \to 0$ as $n \to \infty$ if $r < 1$, and $|x_n| \to \infty$ as $n \to \infty$ if $r > 1$. However, the solution remains bounded, but does not approach zero if $r = 1$.