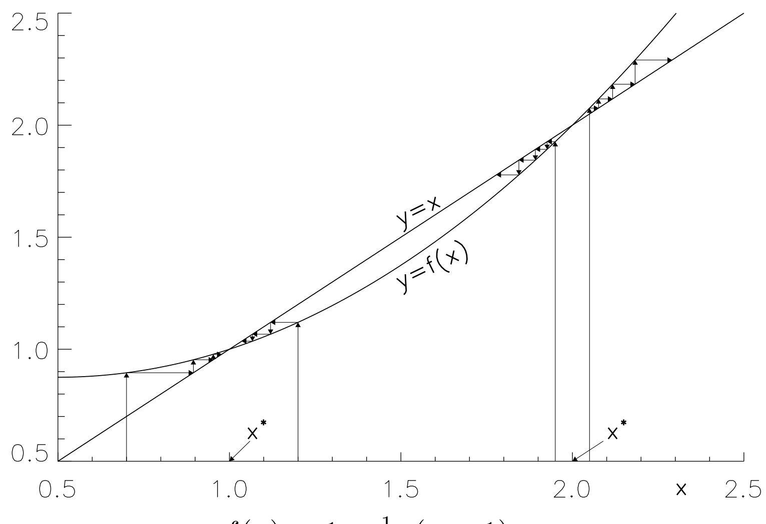
Discrete modelling: Cobweb diagrams, stability, k-periodic point

Consider the first-order dynamical system

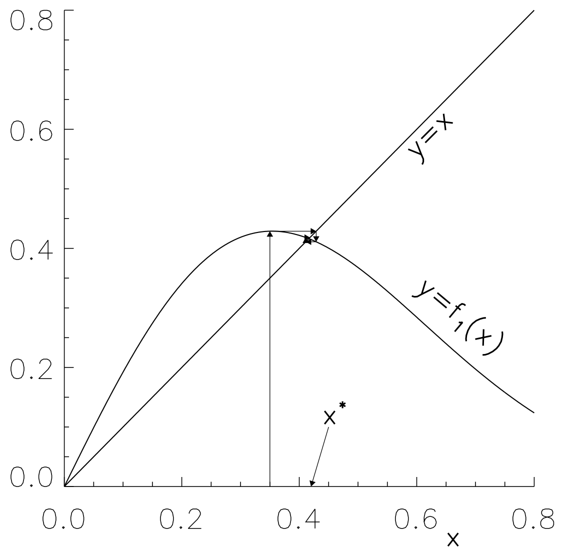
$$x_{n+1} = f(x_n),\ n \geq 0.$$

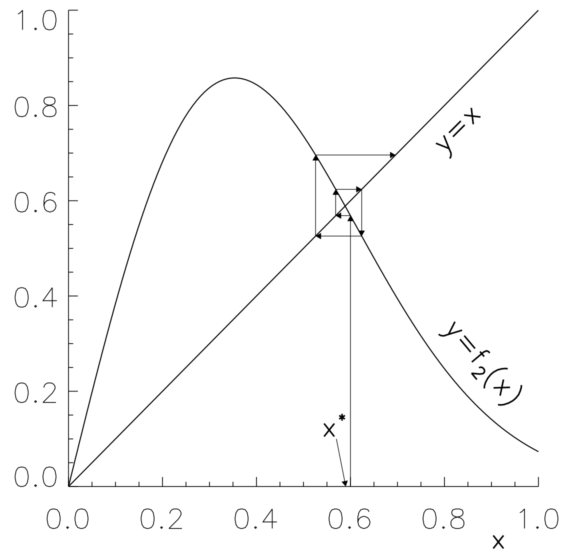
We start with an important graphical method to analyze the stability of equilibrium points. We draw the graph of $y = f(x)$ and $y = x$. The intersection of these two is an equilibrium point. Given the initial point x_0, we locate the value $x_1 = f(x_0)$ by drawing a vertical line through x_0 that intersects the graph of $y = f(x)$ at (x_0, x_1). To locate x_1 on the $x-$axis, we draw a horizontal line from (x_0, x_1) to meet the graph $y = x$ at the point (x_1, x_1). A vertical line drawn from the point (x_1, x_1) will meet the graph of $y = f(x)$ at the point (x_1, x_2), where $x_2 = f(x_1)$. Continuing this process, we find x_n. The figure below shows the cobweb diagrams where $f(x) = 1 + \frac{1}{2}x(x-1)$. Here, the equilibrium points are $x^* = 1$ and $x^* = 2$. The cobweb diagram with x_0 near $x^* = 1$ shows that the iterates converge to $x^* = 1$, and hence we conjecture that $x^* = 1$ is a stable equilibrium point. However, the iterates diverge from $x^* = 2$ when the initial point x_0 is near $x^* = 2$, and hence we conjecture that $x^* = 2$ is an unstable equilibrium point. Also, observe that the convergence to, or divergence from, the equilibrium point is monotonic since $f'(x^*) > 0$.



$f(x) = 1 + \frac{1}{2}x(x-1)$

The figure in the next page considers $x_{n+1} = f_1(x_n)$, $x_n \geq 0$, where $f_1(x) = 2x\mathrm{e}^{-4x^2}$ [see panel (a)] and $x_{n+1} = f_2(x_n)$, $x_n \geq 0$, where $f_2(x) = 4x\mathrm{e}^{-4x^2}$ [see panel (b)]. For f_1, the iterates with initial point close to $x^* \approx 0.42$ converge, and the convergence is oscillatory. Similarly, for f_2, the iterates with initial point close to $x^* \approx 0.59$ diverge in an oscillatory fashion. Also, $f'(x^*) < 0$ causes the oscillatory convergence or divergence from the equilibrium point. We conjecture that x^* for f_1 is stable and x^* for f_2 is unstable. However, to be sure, we need the analytical tools described next.





(a) $f_1(x) = 2x\mathrm{e}^{-4x^2},\ x^* \approx 0.42$ (b) $f_2(x) = 4x\mathrm{e}^{-4x^2},\ x^* \approx 0.59$

Definition: The equilibrium point x^* is stable if for a given $\epsilon > 0$, there exists a $\delta > 0$ such that for any x satisfying $|x - x^*| < \delta$, we have $|f^n(x) - x^*| < \epsilon$ for all $n > 0$. If x^* is not stable, then it is called unstable. Thus, x^* is unstable if there is an $\epsilon > 0$ such that for any $\delta > 0$, there are x and n such that $|x - x^*| < \delta$ and $|f^n(x) - x^*| \geq \epsilon$.

Definition: A fixed point x^* is called attracting if there exists $\eta > 0$ such that $|x - x^*| < \eta$ implies $\lim_{n\to\infty} f^n(x) \to x^*$.

Definition: A fixed point x^* is called an asymptotically stable if it is stable and attracting. If $\eta = \infty$, then x^* is called globally asymptotically stable.

Theorem: Let x^* be a fixed point of the difference equation $x_n = f(x_{n-1})$, $n \geq 1$, where f is continuously differentiable at x^*. If $|f'(x^*)| < 1$, then x^* is asymptotically stable. On the other hand, x^* is unstable when $|f'(x^*)| > 1$.

Proof: Since $f'(x)$ is continuous near x^* and $|f'(x^*)| < 1$, there exists a $\tilde{\epsilon} > 0$ such that for any K with $|f'(x^*)| < K < 1$, we have $|f'(x)| \leq K < 1$ for every $x \in J = (x^* - \tilde{\epsilon}, x^* + \tilde{\epsilon})$. Let x be any point in J. Then

$$f(x) - x^* = f(x) - f(x^*) = f'(\eta)(x - x^*)$$

for some η between x and x^* and hence $\eta \in J$. Then

$$|f(x) - x^*| = |f'(\eta)|\,|(x - x^*)| \leq K|x - x^*| < |x - x^*| < \tilde{\epsilon}.$$

Thus, $f(x)$ lies in J. Also, $f(x)$ lies closer to x^* than x. Using similar argument, since $f(x) \in J$, we find

$$|f^2(x) - x^*| = |f(f(x)) - x^*| \leq K|f(x) - x^*| \leq K^2|x - x^*|$$

Since $K < 1$, we have $|f^2(x) - x^*| < |f(x) - x^*|$ and hence $f^2(x)$ lies closer to x^* than $f(x)$. Continuing this argument, we find that

$$|f^n(x) - x^*| \leq K^n|x - x^*|$$

Choose $\delta = \min\{\tilde{\epsilon}, \epsilon\}$. Then for $|x - x^*| < \delta$, we find

$$|f^n(x) - x^*| \leq K^n |x - x^*| < |x - x^*| < \delta \leq \epsilon$$

Hence, the fixed point x^* is stable. Also, since $K < 1$, from $|f^n(x) - x^*| \leq K^n|x - x^*|$, we conclude that $f^n(x) \to x^*$ as $n \to \infty$. Hence, the fixed point x^* is locally asymptotically stable.

To prove the other part of the theorem, we observe that there is an $\epsilon > 0$ such that $|f'(x)| \geq M > 1$ for $x \in I = (x^* - \epsilon, x^* + \epsilon)$. Take an arbitrary $\delta > 0$ such that $\delta < \epsilon$. Choose $x \in (x^* - \delta, x^* + \delta)$. Then

$$f(x) - x^* = f(x) - f(x^*) = f'(\zeta)(x - x^*),$$

where ζ lies between x and x^* and hence in $I = (x^* - \epsilon, x^* + \epsilon)$. Thus,

$$|f(x) - x^*| = |f'(\zeta)|\,|(x - x^*)| \geq M|x - x^*| > |x - x^*|$$

Thus, the distance between $f(x)$ and x^* is larger than the distance between x and x^*. If $f(x)$ is outside I, then we are done. Otherwise, since $f(x) \in I$, repeating the argument, we find

$$|f^2(x) - x^*| \geq M|f(x) - x^*| \geq M^2|x - x^*|$$

Thus, $f^2(x)$ is further away from x^* than $f(x)$. If it is in I, we continue the procedure till $|f^n(x) - x^*| \geq M^n|x - x^*| > \epsilon$ for some n. This is always possible since $M > 1$.

Definition: (k-periodic point) For a first order dynamical system $x_{n+1} = f(x_n)$, $n \geq 0$, a point $\bar{x}$ is a k-periodic point if $f^k(\bar{x}) = \bar{x}$. Here, f^k denotes the k-th iterate of f (i.e., applying f consecutively k times).The minimal period k is the smallest positive integer for which this relation holds, meaning $f^j(\bar{x}) \neq \bar{x}$ for all $1 \leq j < k$.

Orbit: The set $O(\bar{x}) = \{\bar{x}, f(\bar{x}), f^2(\bar{x}), \ldots, f^{k-1}(\bar{x})\}$ of distinct points generated by a k-periodic point $\bar{x}$ forms a periodic orbit (or cycle) of length k. Note that a 1-periodic point ($k = 1$) satisfies $f(\bar{x}) = \bar{x}$ and is simply called a fixed point. Every k-periodic point of f is a fixed point of the k-iterate map f^k.

Definition: Let $\bar{x}$ be a k-periodic point of f. Then $\bar{x}$ is

i. stable if it is a stable fixed point of f^k;

ii. asymptotically stable if it is an asymptotically stable fixed point of f^k;

iii. unstable if it is an unstable fixed point of f^k.

It follows that if $\bar{x}$ is k-periodic, then every point of its k-cycle $\{x_1 = \bar{x}, x_2 = f(\bar{x}), x_3 = f^2(\bar{x}), \cdots, x_k = f^{k-1}(\bar{x})\}$ as also k-periodic. This follows from

$$f^k(x_r) = f^k\left(f^{r-1}(\bar{x})\right) = f^{r-1}\left(f^k(\bar{x})\right) = f^{r-1}(\bar{x}) = x_r, \qquad r = 1, 2, 3, \cdots, k.$$

Theorem: Let $O(\bar{x}) = \{x_1 = \bar{x}, x_2 = f(\bar{x}), x_3 = f^2(\bar{x}), \cdots, x_k = f^{k-1}(\bar{x})\}$ be a k-cycle of a continuously differentiable function f. Then

i. The k-cycle $O(b)$ is asymptotically stable if

$$|f'(x_1)f'(x_2)\cdots f'(x_k)| < 1$$

ii. The k-cycle $O(b)$ is unstable if

$$|f'(x_1)f'(x_2)\cdots f'(x_k)| > 1$$

Proof: Let us see for $k = 2$. In this case $x_1 = \bar{x}$ and $x_2 = f(\bar{x})$, which also implies that $x_1 = f(x_2)$. Now both x_1 and x_2 are fixed points of $g = f^2$. We know that x_1 is asymptotically stable when $|g'(x_1)| < 1$ and it is unstable when $|g'(x_1)| > 1$. Since $g(x) = f^2(x) = f(f(x))$, we find, by chain rule,

$$g'(x) = f'(f(x))\, f'(x) \implies g'(x_1) = f'(f(x_1))f'(x_1) = f'(x_2)f'(x_1)$$

Similarly, $g'(x_2) = f'(x_1)f'(x_2)$. Hence, the 2-cycle, $\{x_1 = \bar{x}, x_2 = f(\bar{x})\}$, is asymptotically stable when $|f'(x_1)f'(x_2)| < 1$ and unstable when $|f'(x_1)f'(x_2)| > 1$. This result can be generalised to k-cycle.

Discrete logistic equation and bifurcations: Using the idea of limited resources at high population density, we model a discrete population growth model as

$$y_{n+1} - y_n = ry_n\left(1 - \frac{y_n}{K}\right).$$

We write this as

$$y_{n+1} = (1+r)y_n\left(1 - \frac{ry_n}{K(1+r)}\right)$$

Introducing $x_n = ry_n/K(1+r)$ in the above, we find

$$\frac{K(1+r)}{r}x_{n+1} = (1+r)\frac{K(1+r)}{r}x_n(1-x_n) \implies x_{n+1} = (1+r)x_n(1-x_n)$$

Writing $a = 1 + r$, the above becomes

$$x_{n+1} = ax_n(1-x_n) \equiv f(x_n), \quad x_n \in [0,1],$$

which is often called the discrete logistic equation, where $a > 0$ is a parameter. We restrict x_n between $[0,1]$ to ensure the feasibility of the population.

To examine the solution's dependence on the parameter a, we first identify the fixed points and their stability. To find the fixed point, we solve $x = f(x)$, which gives

$$x = ax(1-x) \implies x\,(1 - a(1-x)) = 0 \implies x_1^* = 0,\, x_2^* = 1 - \frac{1}{a}.$$

Here, the second equilibrium point x_2^* exists for $a \geq 1$. It coincides with $x_1^* = 0$ for $a = 1$. Now $f'(x) = a(1-2x)$ and hence

$$f'(x_1^*) = a, \quad f'(x_2^*) = a - 2a\left(1 - \frac{1}{a}\right) = 2 - a$$

Hence, $x_1^* = 0$ is stable when $|a| < 1 \implies -1 < a < 1$. Since $a > 0$, this implies $0 < a < 1$. Similarly, it is unstable when $|a| > 1 \implies a < -1$ or $a > 1$. Since $a > 0$, this implies $a > 1$. On the other hand, $|2-a| < 1 \implies 1 < a < 3$ and $|2-a| > 1 \implies a > 3$ or $a < 1$. Since x_2^* is feasible for $a \geq 1$, we conclude that $x_2^* = 1 - \frac{1}{a}$ is stable for $1 < a < 3$ and unstable for $a > 3$.

For 2-periodic orbit, we solve $f^2(x) = x$, i.e.,

$$af(x)\big(1 - f(x)\big) = x \implies a^2x(1-x)\big(1 - ax(1-x)\big) - x = 0.$$

Since fixed points (i.e., 1-periodic points) are also solutions of the above equations, x and $x - 1 + \frac{1}{a}$, i.e., x and $ax - a + 1$ must be factors of the above equation. The above equation is written as

$$x\big(a^3x^3 - 2a^3x^2 + a^2(a+1)x - (a^2-1)\big) = 0 \implies x(ax-a+1)\big(a^2x^2 - (a^2+a)x + (a+1)\big) = 0$$

Hence, 2-periodic points x_1^p and x_2^p are the solutions of

$$a^2x^2 - (a^2+a)x + (a+1) = 0.$$

This implies

$$x_1^p + x_2^p = \frac{a+1}{a}, \; x_1^p x_2^p = \frac{a+1}{a^2}.$$

We also need to show that x_1^p and x_2^p lie in the interval $[0, 1]$. Solving, we find

$$x_1^p = \frac{(a+1) - \sqrt{(a-3)(a+1)}}{2a}, \quad x_2^p = \frac{(a+1) + \sqrt{(a-3)(a+1)}}{2a},$$

which exist only when $a > 3$. Now $a - 3 < a + 1$ and hence

$$x_1^p = \frac{(a+1) - \sqrt{(a-3)(a+1)}}{2a} > \frac{(a+1) - \sqrt{(a+1)^2}}{2a} = 0$$

Further,

$$x_2^p = \frac{(a+1) + \sqrt{(a-1-2)(a-1+2)}}{2a} = \frac{(a+1) + \sqrt{(a-1)^2 - 4}}{2a} < \frac{(a+1) + \sqrt{(a-1)^2}}{2a} = 1$$

Hence, we have found that $0 < x_1^p < x_2^p < 1$ for $a > 3$. For the stability of the 2-periodic cycle, we need to show that $|f'(x_1^p)f'(x_2^p)| < 1$. Now,

$$\begin{aligned} f'(x_1^p)f'(x_2^p) &= a^2(1 - 2x_1^p)(1 - 2x_2^p) \\ &= a^2\big(1 - 2(x_1^p + x_2^p) + 4x_1^p x_2^p\big) \\ &= a^2\left(1 - 2\frac{a+1}{a} + 4\frac{a+1}{a^2}\right) \\ &= 5 - (a-1)^2 \end{aligned}$$

Thus, $|f'(x_1^p)f'(x_2^p)| < 1$, when

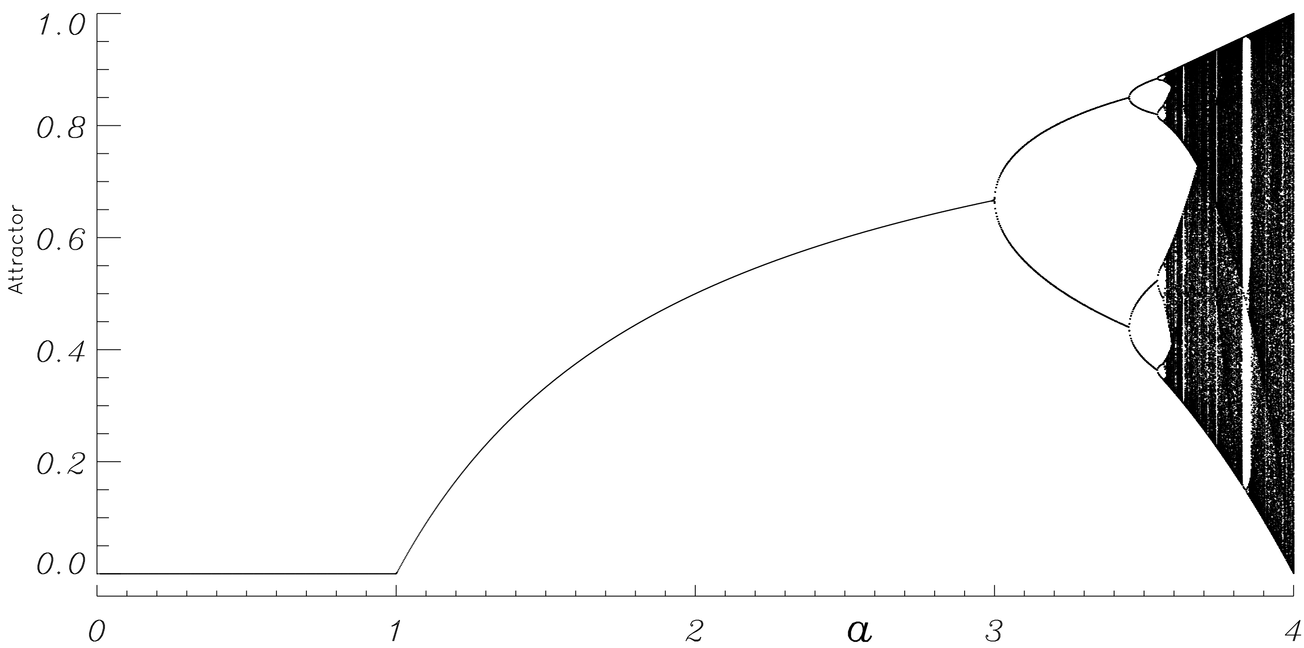
$$\begin{aligned} |5 - (a-1)^2| < 1 &\implies 4 < (a-1)^2 < 6 \implies 2 < |a-1| < \sqrt{6} \\ |a-1| > 2 &\implies a > 3 \text{ or } a < -1 \implies a > 3 \qquad \text{since } a > 0, \\ |a-1| < \sqrt{6} &\implies 1 - \sqrt{6} < a < 1 + \sqrt{6}. \end{aligned}$$

Combining these two cases, we conclude that the 2-periodic points x_1^p and x_2^p are stable when $3 < a < 1 + \sqrt{6}$. Also,

$$|5 - (a-1)^2| > 1 \implies |(a-1)| < 2 \text{ and } |a-1| > \sqrt{6}$$
$$\implies 1 < a < 3 \text{ and } (a > 1 + \sqrt{6} \text{ or } a < 1 - \sqrt{6}).$$

Since the 2-periodic point exists for $a > 3$, we conclude that x_1^p and x_2^p are unstable when $a > 1 + \sqrt{6}$.

A summary of our results so far is as follows: $x^* = 0$ is the stable equilibrium point for $0 < a < 1$, followed by $x^* = 1 - \frac{1}{a}$, which is stable for $1 < a < 3$. We have a stable 2-periodic solution for $3 < p < 1 + \sqrt{6}$. As r increases beyond $1 + \sqrt{6}$, we get a stable 4-periodic cycle. This period-doubling episode continues, yielding 8-, 16- and so on until $r = r_\infty \approx 3.57$. The map becomes chaotic beyond r_∞, and the attractor (stable set) changes from a finite to an infinite set of points. For $r > r_\infty$, the orbit diagram reveals a fascinating mixture of order and chaos, with periodic windows interspersed between chaotic clouds of dots. The large window appearing near $r \approx 3.83$ contains a stable period-3 cycle. A zoom-up of any part of the period-3 window would show that a copy of the orbit diagram reappears.



Bifurcation diagram of the Logistic map $x_{n+1} = a x_n (1 - x_n)$