Dimensional Analysis I

One of the basic techniques that is useful in the mathematical modelling of a problem is the analysis of the relevant quantities that influence the model and how they must relate to each other in a dimensional way. An important property is that equations representing physical laws must be dimensionally homogeneous. If we have an equation that is a sum of many terms, then each of these terms must have the same dimensions. This idea forms the basis of dimensional analysis. Dimensional analysis has enabled the determination of physical phenomena even when the governing equations are unknown.

Dimensional homogeneity: Consider

$$s = \frac{1}{2}gt^2,$$

which gives the distance s an object will fall when released from rest at $t = 0$ in a constant gravity field (air resistance negligible). Note that the dimensions of distance and time are L and T, respectively. We use $[q]$ to denote the dimension of the physical quantity q. Thus, $[s] = L$ and $[t] = T$. Since g is the acceleration due to gravity, its dimension is $[g] = LT^{-2}$. We see that the equation is dimensionally homogeneous.

Once an equation is dimensionally homogeneous, its form remains unchanged when expressed in different units. Sometimes this is called unit-independent or unit-free. For example, consider the equation given above, which was derived from the laws of physics. If we measure distance in metres and time in seconds, then g is approximately 9.81 m/s^2. Suppose we want to measure distance in kilometres and time in hours. Further, we use bar $(-)$ for the new units. Then

$$\bar{s} = \lambda_1 s \quad \text{and} \quad \bar{t} = \lambda_2 t,$$

where $\lambda_1 = \frac{1}{1000}$ (km/m) and $\lambda_2 = \frac{1}{3600}$ (h/s). Also, since acceleration due to gravity has dimension LT^{-2}, we have

$$\bar{g} = g\lambda_1\,\lambda_2^{-2}$$

Substituting these into $s = \frac{1}{2}gt^2$, we find

$$\frac{\bar{s}}{\lambda_1} = \frac{1}{2}\frac{\bar{g}}{\lambda_1\,\lambda_2^{-2}}\frac{\bar{t}^2}{\lambda_2^2} \implies \bar{s} = \frac{1}{2}\bar{g}\bar{t}^2$$

Thus, the equation remains the same when we express the dimensional variables in new units.

Buckingham's Pi theorem: Suppose we have a unit-independent physical law in the form

$$f(q_1, q_2, \cdots, q_n) = 0,$$

where the q_i are dimensional variables. Further, suppose we can form m new independent dimensionless variables $\Pi_1, \Pi_2, \cdots, \Pi_m$ $(m \le n)$ by combining q_i. Then, $f(q_1, q_2, \cdots, q_n) = 0$

is equivalent to a physical law of the form

$$h(\Pi_1, \Pi_2, \cdots, \Pi_m) = 0$$

for some (unknown) function h. We will outline a proof of the theorem later.

Remark: It may happen that the physical law also depends on dimensionless parameters $r_1, r_2, \cdots, r_k$. These may include angles in radians, strain, and other measures. In that case, the physical law becomes $f(q_1, q_2, \cdots, q_n, r_1, r_2, \cdots, r_k) = 0$ and the Pi theorem implies a function $h(\Pi_1, \Pi_2, \cdots, \Pi_m, r_1, r_2, \cdots, r_k) = 0$.

Example 1: Consider the problem of finding the distance s an object will fall, when released from rest, under the action of gravity at time t. Assume that the air resistance is negligible. We suppose that the relation will involve dimensional variables s, t and g, where g is the acceleration due to gravity. Thus, our physical law is of the form $f(s, t, g) = 0$. Now, let us find dimensionless variables from these dimensional variables. We write

$$\Pi = s^a t^b g^c,$$

where the exponents a, b and c are such that Π is dimensionless. We use M, L and T to denote the fundamental dimensions of mass, length and time, respectively. Then, $[s] = L$, $[t] = T$, $[g] = LT^{-2}$, and $[\Pi] = 1$, since Π is dimensionless. Now, we must have

$$1 = [\Pi] = [s]^a [t]^b [g]^c = L^a T^b (LT^{-2})^c = L^{a+c} T^{b-2c}$$
$$\implies a + c = 0,\ b - 2c = 0 \implies c = -a,\ b = -2a$$

Thus,

$$\Pi = \left(\frac{s}{gt^2}\right)^a$$

Hence, we have only one independent non-dimensional parameter $\Pi = s/gt^2$, and from the Buckingham Pi theorem, we conclude that

$$h\left(\frac{s}{gt^2}\right) = 0 \implies \frac{s}{gt^2} = k, \qquad \text{where } k \text{ is a constant}$$

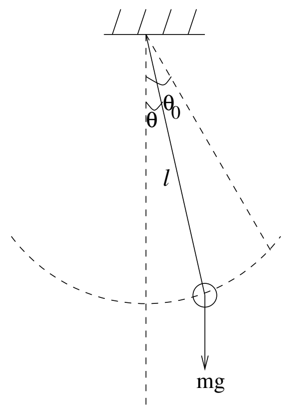
Note that k is a root of $h(x) = 0$, and k is unknown. Rearranging the above, we find

$$s = kgt^2.$$

From physics, we know that $k = \frac{1}{2}$. In case we don't have a value, it can be estimated from an experiment if required.

Example 2 : Consider an experiment in which a pendulum bob, of mass m and length l, is released from rest from an angle θ_0. Here, θ_0 is in radians, and hence it is dimensionless. It will oscillate, and let τ be the time period of oscillation. We need to decide how τ depends on various parameters. We assume that the resistance force is negligible. Obviously, τ depends on l, m, g and θ_0. Let $f(\tau, l, m, g, \theta_0) = 0$. Since, θ_0 is dimensionless, we construct

$$\Pi = \tau^a l^b m^c g^d \implies [\Pi] = [\tau]^a [l]^b [m]^c [g]^d = T^a L^b M^c (LT^{-2})^d = T^{a-2d} L^{b+d} M^c$$



Since $[\Pi] = 1$, we must have

$$a - 2d = b + d = c = 0 \implies d = \frac{a}{2},\ b = -\frac{a}{2},\ c = 0 \implies \Pi = \left(\tau\sqrt{\frac{g}{l}}\right)^a$$

Since we want τ in terms of other parameters, we solve the other exponents in terms of the exponent a of τ.. Hence, we get two dimensionless parameters $\Pi_1 = \tau\sqrt{\frac{g}{l}}$ and $\Pi_2 = \theta_0$. Thus, $f(\tau, l, m, g, \theta_0) = 0$ becomes $g(\Pi_1, \Pi_2) = 0$ or

$$\Pi_1 = h(\Pi_2) \implies \tau\sqrt{\frac{g}{l}} = h(\theta_0) \implies \tau = \sqrt{\frac{l}{g}}h(\theta_0).$$

We don't know the function h but we can infer about many other things.

a. If everything remain same except that the length of pendulum string is doubled. Then

$$\frac{\tau_{2l}}{\tau_l} = \sqrt{2} \implies \tau_{2l} = \sqrt{2}\tau_L$$

Hence, the time period of the pendulum of twice the length is $\sqrt{2}$ times the time period of the original pendulum.

b. We know that the g_m on the Moon is approximately $\frac{1}{3}$-rd of g of Earth. Hence,

$$\frac{\tau_M}{\tau_E} = \sqrt{3} \implies \tau_M = \sqrt{3}\tau_E$$

Hence, the time period of the pendulum on the Moon is $\sqrt{3}$ times that of Earth.

c. Clearly, $h(\theta_0)$ is an even function since releasing from θ_0 or $-\theta_0$ lead to same time period. If θ_0 is small, then we can express h in a power series about $\theta_0 = 0$:

$$h(\theta_0) = h(0) + \frac{\theta_0^2}{2!}h''(0) + \cdots$$

Since θ_0 is small, $h(\theta_0) \approx h(0) = k$, a constant. In this case, we get

$$\tau = k\sqrt{\frac{l}{g}},$$

where k is again unknown. It will be shown below that $k = 2\pi$ when θ_0 is small.

d. Note that the arc length $s = l\theta$ and hence tangential velocity and acceleration (in the increasing θ direction) are $l\theta'$ and $l\theta''$, respectively. Resolving the gravitational force along the tangential direction, we find the equation of motion as

$$ml\theta'' = -mg\sin\theta \implies \theta'' + \frac{g}{l}\sin\theta = 0, \qquad \theta(0) = \theta_0,\ \theta'(0) = 0.$$

From this, we could have concluded that $f(\tau, l, g, \theta_0) = 0$.

e. If θ_0 is small, then θ is also small and hence $\sin\theta \approx \theta$. Then, the governing equation becomes

$$\theta'' + \frac{g}{l}\theta = 0, \qquad \theta(0) = \theta_0,\ \theta'(0) = 0.$$

Its solution is $\theta = \theta_0 \cos\sqrt{\frac{g}{l}}t$, whose time period τ satisfies

$$\theta(t+\tau) = \theta(t) \implies \theta_0\cos\sqrt{\frac{g}{l}}(t+\tau) = \theta_0\cos\sqrt{\frac{g}{l}}t \implies \tau = 2\pi\sqrt{\frac{l}{g}}.$$

Hence, $k = 2\pi$ in the point c above when θ_0 is small.