

Optimization I

An optimization can be formulated as

$$\begin{aligned} &\text{Optimize } f_j(X) \quad \text{for } j \in J \\ &\text{subject to} \\ &g_i(X) \leq b_i \quad \text{for all } i \in I, \end{aligned}$$

where I and J are some index sets. The components of the vector X are called decision variables, and $f_j(X)$ are called objective functions. Note that there may be more than one objective function. The side conditions $g_i(X) \leq b_i$ are called constraints. We seek the vector X_0 that optimises the objective functions and simultaneously satisfies each constraint. There exist two types of optimization: unconstrained and constrained.

Unconstrained optimization: First, consider a problem in which we optimise $f(x)$, where f is a function of a single variable x . To solve it, we proceed as follows:

- i. Find the first derivative $f'(x)$ of the function and set it to zero to find the stationary or critical points.
- ii. Check the maxima/minima by the second derivative test or sign change of the first derivative.
- iii. Substitute the x values back into the original function $f(x)$ to get the actual maximum or minimum values.

Second, we consider an unconstrained problem in which we optimise $f(x, y)$, where f is a function of two variables x and y . To solve it, we proceed as follows:

- i. Find the first partial derivatives f_x and f_y of the function f with respect to x and y . Set both derivatives equal to zero to form a system of equations:

$$f_x(x, y) = 0 \quad \text{and} \quad f_y(x, y) = 0$$

Solve these equations simultaneously to find the coordinates (a, b) of the critical (stationary) points.

- ii. Calculate second-order partial derivatives at each critical point (a, b) :

$$A = \frac{\partial^2 f}{\partial x^2}, \quad B = \frac{\partial^2 f}{\partial x \partial y}, \quad C = \frac{\partial^2 f}{\partial y^2}$$

iii. Calculate the discriminant D for each critical point (a, b) :

$$D = AC - B^2$$

Use the value of D and A to classify the critical point (a, b) : a local minimum if $D > 0$ and $A > 0$; a local maximum if $D > 0$ and $A < 0$; a saddle point if $D < 0$; and inconclusive (test fails) if $D = 0$. We need further investigation into the case when $D = 0$.

iv. Substitute the coordinates of the local maximum or minimum points back into the original function $f(x, y)$ to find the actual maximum or minimum value.

Constrained optimization: Suppose we want to optimize $f(x, y)$, a function of two variables, with one equality constraint. We solve the equality constraint for one variable (say, y) and substitute into the objective function f . Now it becomes an unconstrained optimization problem for a single variable.

Example: Consider the problem: maximize xy subject to $x + y = 4$. From the equality constraint, we find $y = 4 - x$. Substituting into the objective function, the problem becomes maximize $g(x) = x(4 - x)$. Now $g'(x) = 0$ implies $x = 2$ and $g''(x) = -2$ implies $x = 2$ gives maximum. Thus, the maximum value is 4.

If the objective function or the equality constraint contains more than two variables, then we use the Lagrange multiplier method. For example, consider the problem: optimize $f(x_1, x_2, \dots, x_n)$ subject to $g(x_1, x_2, \dots, x_n) = 0$. We introduce Lagrange multiplier λ and construct $F = f - \lambda g$. Next, find all the critical points of $(x_1, x_2, \dots, x_n, \lambda)$ of F . Then, evaluate f at all these critical points. The largest and smallest values are the maximum and minimum, respectively.

Example: We find the maximum volume of a box whose surface area is 54 square units. Suppose x, y and z are the length, width and height of the box. Our problem is to maximize xyz subject to $2(xy + yz + zx) = 54$. Let $f(x, y, z) = xyz$ and $g(x, y, z) = xy + yz + zx - 27$. Construct $F = f - \lambda g$. Now $F_x = F_y = F_z = F_\lambda = 0$ imply

$$yz - \lambda(y + z) = 0, \quad xz - \lambda(x + z) = 0, \quad xy - \lambda(x + y) = 0, \quad xy + yz + zx - 27 = 0$$

The first two relations gives $\lambda x(y + z) = \lambda y(x + z) \implies \lambda z(x - y) = 0$. Hence, $x = y$ since $\lambda \neq 0$ and $z \neq 0$. (Note that if $\lambda = 0$, then the constraint $g = 0$ becomes independent of f and the critical points of f must satisfy $g = 0$. Now critical points of f are solutions of $f_x = f_y = f_z = 0 \implies yz = zx = xy = 0 \implies x = y = z = 0$, which do not satisfy $g = 0$. Hence, λ must be nonzero.) From the third relation, we find $x(x - 2\lambda) = 0 \implies x = 2\lambda$. Using the second, we get $2z - x - z = 0 \implies x = z$. Hence, $x = y = z$ and using the last relation, we find $x = y = z = 3$. This gives a volume of 27. If we use $x = y = 1$, then $z = 13$ satisfies the constraint, and this gives volume $13 < 27$. Hence, the volume 27 with $x = y = z = 3$ is indeed the maximum.

If there are two constraints $g(x_1, x_2, \dots, x_n) = 0$ and $h(x_1, x_2, \dots, x_n) = 0$, then introduce Lagrange multipliers λ and μ and construct $F = f - \lambda g - \mu h$. Next, find all the critical points

of $(x_1, x_2, \dots, x_n, \lambda, \mu)$ of F . Then, evaluate f at all these critical points. The largest and smallest values are the maximum and minimum, respectively.

Example: We find the maximum volume of a box whose surface area is 54 square units and base area is 6 square units. If x, y, z are the length, width and height, then our problem is to maximize xyz subject to $xy + yz + zx = 27$ and $xy = 6$. Here, $f(x, y, z) = xyz$, $g(x, y, z) = xy + yz + zx - 27$ and $h(x, y, z) = xy - 6$. We could have solved x or y from $xy = 6$ and then substituted into f and g . This would have resulted in an optimization problem with two variables and one equality constraint, and consequently only one Lagrange multiplier. However, we solve it here using two Lagrange multipliers.

Construct $F(x, y, z, \lambda, \mu) = f - \lambda g - \mu h$. Now, $F_x = F_y = F_z = F_\lambda = F_\mu = 0$ imply

$$\begin{aligned} yz - \lambda(y + z) - \mu y &= 0, \quad xz - \lambda(x + z) - \mu x = 0, \quad xy - \lambda(x + y) = 0, \\ xy + yz + zx - 27 &= 0, \quad xy - 6 = 0 \end{aligned}$$

Using $xy = 6$ in the third equation, we find $\lambda(x + y) = 6$ which implies that $\lambda \neq 0$. Now, multiply first and second relations by x and y , we find

$$xyz - \lambda x(y + z) - \mu xy = 0, \quad xyz - \lambda y(x + z) - \mu xy = 0 \implies \lambda z(x - y) = 0 \implies x = y,$$

since $z \neq 0, \lambda \neq 0$. Using $x = y$ in $xy = 6$, we find $x = y = \sqrt{6}$. Further, using $xy + yz + zx = 27$, we get $z = \frac{21}{2\sqrt{6}}$. Hence, the volume becomes $\frac{21\sqrt{6}}{2}$.

Take $x = 2$ and $y = 3$ that satisfy the second constraint. Using $z = \frac{21}{5}$ satisfies the first constraint and the volume becomes $\frac{126}{5}$ which is less than $\frac{21\sqrt{6}}{2}$. Thus, the maximum volume is $\frac{21\sqrt{6}}{2}$.

Example: Consider a carpenter who makes a profit of ₹250 per chair and ₹300 per table. The carpenter needs 20 square units of board material and 5 hours of labour to make a chair, and 30 square units of board material and 4 hours of labour to make a table. There are 690 square units of board material and 120 hours of labour available per week. Furthermore, at least 4 chairs and 2 tables must be made per week. Find the number of units for chairs and tables to maximize the profit (non-integer values of chairs and tables are allowed).

To solve it, suppose x_1 and x_2 are the number of chairs and tables, respectively. Then the optimization problem becomes

$$\begin{aligned} \text{Maximize} \quad & z = 250x_1 + 300x_2 \\ \text{subject to} \quad & 20x_1 + 30x_2 \leq 690, \\ & 5x_1 + 4x_2 \leq 120, \\ & x_1 \geq 4, \\ & x_2 \geq 2. \end{aligned}$$

The above problem is an example of a linear programming problem (LPP). Generally, linear programming is a technique for the optimization of a linear objective function, subject to linear

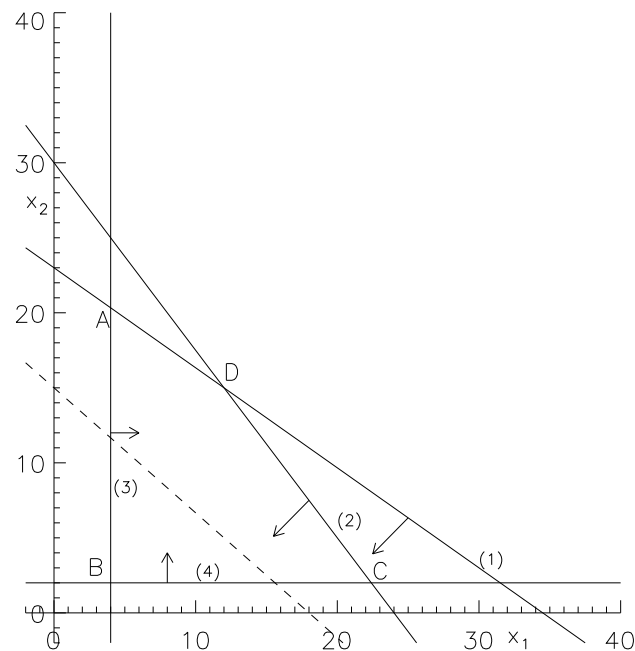
equality and linear inequality constraints. The feasible region in a linear programming problem is the set of all possible points or variable values that satisfy every constraint.

Corner Point Theorem: If a linear programming problem has an optimal value (maximum or minimum), it must occur at a corner point (vertex) of the feasible region.

Bounded Region theorem: If the feasible region is bounded, the objective function has both a maximum and a minimum value, and both occur at the corner points.

Further, if the feasible region is open or unbounded, a maximum or minimum value may not exist. But if an optimal value does exist, it still occurs at a corner point. In addition, if the optimal value occurs at two different corner points, it also occurs at every point on the line segment connecting them, which implies that multiple solutions to an LPP may exist.

Example: Let us solve the LPP associated with maximizing the profit of the carpenter. We can readily solve it graphically.



The straight lines corresponding to constraints are numbered in the order they appeared. Clearly, the feasible region is the closed polygon ABCD. For a profit of ₹ c , we have $z = 250x_1 + 300x_2 = c$. The profit line with $c = 9000$ is shown by the dashed line. Note that changing c implies a parallel line to the drawn dashed line. Clearly, we can increase profit by moving the profit line to the right. It appears that the profit line leaves the feasible region at D. The coordinates of D are (12, 15) and hence $z = c = 250 \times 12 + 300 \times 15 = ₹7500$. Thus, the maximum profit of ₹7500 can be achieved with 12 chairs and 15 tables.

Note that if it is not clear through which corner the profit line leaves the feasible region, we construct a table of corner points and their profits. Since the optimal value must occur at a corner point of the bounded feasible region, we take the corner point(s) corresponding to the maximum profit for the optimal solution. The table verifies the same profit obtained earlier.

Corner points	Coordinates	Profit (z)
A	$(4, \frac{61}{3})$	₹7100
B	$(4, 2)$	₹1600
C	$(\frac{112}{5}, 2)$	₹6200
D	$(12, 15)$	₹7500 ✓

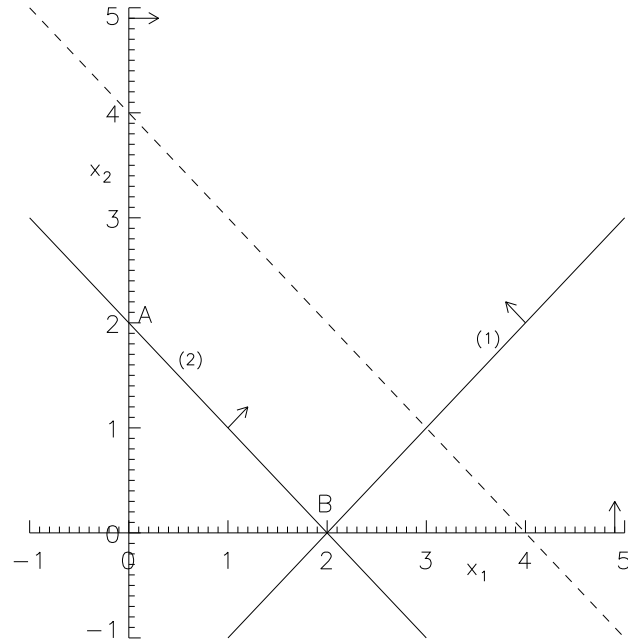
Example (empty and unbounded solution): If the feasible region is empty, then no solution exists. For example, if we have both constraints $x_1 \leq 3$ and $x_2 \geq 5$ in an LPP, then no feasible solution exists. On the other hand, consider the following LPP:

$$\text{Maximize} \quad z = 2x_1 + 2x_2 \quad (1)$$

$$\text{subject to} \quad x_1 - x_2 \leq 2, \quad (2)$$

$$x_1 + x_2 \geq 2, \quad (3)$$

$$x_1 \geq 0, x_2 \geq 0. \quad (4)$$



The feasible region in the first quadrant is bounded by the line segment AB, the line corresponding to constraint (1), and the part of the x_2 -axis above point A. The profit line for $z = 8$ is also shown. Obviously, the profit can be increased by shifting the profit line away from the line corresponding to the constraint (2). Since the feasible region is unbounded, the optimal solution is unbounded.

Suppose we change the objective function to Minimize $z = 2x_1 + 2x_2$. The feasible region remains unchanged. But we now shift the $z = \text{constant}$ line toward the line corresponding to the constraint (2). The $z = \text{constant}$ line leaves the feasible region along the line segment AB. Clearly, the minimum of z will be $z = 4$, which occurs at the corner points A(0, 2) and B(2, 0). Also, any point on the line segment AB has coordinates $(2\lambda, 2(1 - \lambda))$, $0 \leq \lambda \leq 1$. The value of

the objective function on the line segment is $z = 4\lambda + 4(1 - \lambda) = 4$, which is also the optimum value. Hence, the optimum value occurs at two corner points and on the line segment joining them.

Since we know that the optimal solution occurs at a corner point of the feasible region, we just need to find the corner points and choose the best one. To illustrate, consider

$$\begin{aligned} \text{Maximize} \quad & z = 3x_1 + 4x_2 \\ \text{subject to} \quad & 3x_1 + 2x_2 \leq 20, \\ & x_1 + 2x_2 \leq 10, \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

To find the corner points, we first find all the intersection points of the constraint lines. To do this, we introduce two slack variables $s_1 \geq 0, s_2 \geq 0$:

$$\begin{aligned} \text{Maximize} \quad & z = 3x_1 + 4x_2 \\ \text{subject to} \quad & 3x_1 + 2x_2 + s_1 = 20, \\ & x_1 + 2x_2 + s_2 = 10, \\ & x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0. \end{aligned}$$

For any points on the first constraint line, $3x_1 + 2x_2 = 20$, we must have $s_1 = 0$. Similarly, s_2 must be zero on the second constraint line. To find the intersection of the first and second constraint lines, we must put $s_1 = s_2 = 0 \implies x_1 = 5, x_2 = \frac{5}{2}$. The x_1 -axis is the constraint line for $x_2 \geq 0$ and any point on this line has $x_2 = 0$. The intersection of the constraint line $x_2 \geq 0$, and the first constraint is found by solving $3x_1 = 20 \implies x_1 = \frac{20}{3}$. We also need to check the value of the slack variable s_2 at this point to ensure it lies in the feasible region. Substituting $x_1 = \frac{20}{3}, x_2 = 0$ and $s_1 = 0$ in the second constraint line, we find that $s_2 = \frac{20}{3} > 0$. Hence, it is a valid corner point. To find the points of intersection of the constraint lines, we need to set two variables to zero and solve for the other variables. The number of ways this can be done is ${}^4C_2 = 6$. However, not all of these are feasible, as some may not satisfy the non-negativity conditions. We make a table for the above problem to illustrate these points.

Points	x_1	x_2	s_1	s_2	Feasible	z
1	0	0	20	10	Yes	0
2	0	10	0	-10	No	—
3	0	5	10	0	Yes	20
4	$\frac{20}{3}$	0	0	$\frac{10}{3}$	Yes	20
5	10	0	-10	0	No	—
6	5	$\frac{5}{2}$	0	0	Yes	25 ✓