

Optimization II

The simplex method: We first define an LPP for maximization in standard form. An LPP for maximizing objective function is in standard form if it seeks to maximize the objective function $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2, \\ \vdots &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m, \end{aligned}$$

where $x_i \geq 0$ ($i = 1, 2, \dots, n$), and $b_i \geq 0$ ($i = 1, 2, \dots, m$). This has n decision variables and m ($\leq$) constraints. *An LPP for maximization that is not in standard form can also be solved but we shall not pursue those techniques here.* After adding m slack variables, the corresponding system of constraint equations becomes

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + s_1 &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + s_2 &= b_2, \\ \vdots &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + s_m &= b_m, \end{aligned}$$

where $s_i \geq 0$ for $i = 1, 2, \dots, m$. Thus, the total number of variables is $m + n$. Points of intersection of the constraint lines are obtained by setting n variables to zero and solving the m equations in m variables. The number of ways in which to set n variables to zero is ${}^{m+n}C_n = \frac{(m+n)!}{m!n!}$. For an LPP with 25 decision variables and 50 constraints, this becomes $\frac{75!}{25!50!} = 52,588,547,141,148,893,628 \approx 5.25 \times 10^{19}$ which is a huge number of points for a computer to check. Note that solving a linear system of 50 equations in 50 variables for each possible corner point is itself CPU-intensive. Thus, we need a better algorithm for such problems. One such algorithm is the simplex method.

Before discussing the simplex method, we present another practical LPP problem in which we are interested in minimizing the objective function. Suppose a person needs 84 units of vitamin X and 120 units of vitamin Y. Assume that overdosing on vitamin X and Y is harmless. Per unit of a vegetable A contains 10 units of X and 8 units of Y, and per unit of vegetable B contains 2 units of X and 4 units of Y. However, both A and B contain a harmful mineral W and each unit of A and B contain 3 and 4 units of W, respectively. How many units of A

and B should be taken so that the requirement of X and Y is met and the amount of harmful mineral is minimized?

Let x_1 units of A and x_2 units of B be taken. Then, the problem becomes

$$\begin{array}{ll}\text{Minimize} & z = 3x_1 + 4x_2 \\ \text{subject to} & 10x_1 + 2x_2 \geq 84, \\ & 8x_1 + 4x_2 \geq 120, \\ & x_1 \geq 0, x_2 \geq 0.\end{array}$$

This can be solved easily using graphical analysis. We can also use the corner points technique. To make the $(\geq)$ inequalities into equalities, we use surplus variables:

$$\begin{array}{ll}\text{Minimize} & z = 3x_1 + 4x_2 \\ \text{subject to} & 10x_1 + 2x_2 - s_1 = 84, \\ & 8x_1 + 4x_2 - s_2 = 120, \\ & x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0.\end{array}$$

Here, s_1 and s_2 are surplus variables. The other details follow the analysis similar to that discussed earlier for the maximization problem.

The simplex method: A basic solution of an LPP for maximization in standard form (introduced earlier) is a solution of the constraint equations in which at most m variables are nonzero, and a basic solution for which all variables are nonnegative is called a basic feasible solution. The simplex method is carried out by performing elementary row operations on a matrix called the simplex tableau. This tableau consists of the augmented matrix corresponding to the constraint equations together with the coefficients of the objective function written in the form

$$-c_1x_1 - c_2x_2 - \cdots - c_nx_n + 0 \cdot s_1 + 0 \cdot s_2 + \cdots + 0 \cdot s_m + z = 0$$

Usually, the coefficient of z is omitted in a simplex tableau. The simplex algorithm begins at one corner and checks if it gives the optimum value. If the current corner is not the optimum, it moves along the edge to a neighbouring corner that improves the result. It repeats this process until no neighbouring corner offers a better result, meaning the optimum solution is found. Let us first solve an LPP for maximization in standard form with two decision variables using the simplex method:

$$\begin{array}{ll}\text{Maximize} & z = x_1 + 3x_2 \\ \text{subject to} & x_1 + x_2 \leq 4, \\ & -2x_1 + x_2 \leq 1, \\ & 5x_1 - x_2 \leq 8, \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.\end{array}$$

Using slack variables, we write the constraint as

$$\begin{aligned}x_1 + x_2 + s_1 &= 4, \\-2x_1 + x_2 + s_2 &= 1, \\5x_1 - x_2 + s_3 &= 8, \\x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0.\end{aligned}$$

We construct the first simple tableau as follows:

x_1	x_2	s_1	s_2	s_3	b	Basic variables
1	1	1	0	0	4	s_1
-2	1	0	1	0	1	s_2
5	-1	0	0	1	8	s_3
-1	-3	0	0	0	0	←Current z-value

For the initial tableau, the basic variables are s_1, s_2 and s_3 and the non-basic variables are x_1 and x_2 . The non-basic variables have a value zero leading to zero z -value. The b column contains the values of the basic variables $s_1 = 4, s_2 = 2$ and $s_3 = 8$. Thus, the current solution is $x_1 = 0, x_2 = 0, s_1 = 4, s_2 = 1, s_3 = 8$.

Note that the bottom-row entries under x_1 and x_2 are the negatives of the coefficients x_1 and x_2 of the objective function z . To check the optimality of the current solution represented by a simplex tableau, observe the entries in the bottom row of the tableau. If any of these entries are negative (as above), then the current solution is not optimal. Obviously, the current solution is not optimal. To improve the current solution, we bring a new basic variable into the solution, the entering variable. Consequently, one of the current basic variables must leave, the departing variable. Otherwise, we have too many variables for a basic solution. The entering and departing variables are chosen as follows:

- i. The entering variable corresponds to the smallest (the most negative) entry in the bottom row of the tableau.
- ii. The departing variable corresponds to the smallest nonnegative ratio of b_i/a_{ij} in the column determined by the entering variable.
- iii. The entry in the simplex tableau in the entering variable's column and the departing variable's row is called the pivot.

Finally, to form the improved solution, we apply Gauss elimination to the column that contains the pivot, and the process is called pivoting. To improve this solution, we find that x_2 is the entering variable since -6 is the most negative in the bottom row. Now we calculate the non-negative ratios $\frac{4}{1} = 4$ and $\frac{1}{1} = 1$. This implies that s_2 is the departing variable.

x_1	x_2	s_1	s_2	s_3	b	Basic variables	
1	1	1	0	0	4	s_1	
-2	①	0	1	0	1	s_2	$\leftarrow$ departing
5	-1	0	0	1	8	s_3	
-1	-3	0	0	0	0	$\leftarrow$ Current z-value	
	$\uparrow$						
	entering						

With the help of the pivot element, we make the other elements zero in the column for x_2 by the row operations $R_1 \rightarrow R_1 - R_2$, $R_3 \rightarrow R_3 + R_2$, and $R_4 \rightarrow R_4 + 3R_2$.

x_1	x_2	s_1	s_2	s_3	b	Basic variables	
3	0	1	-1	0	3	s_1	
-2	1	0	1	0	1	x_2	
3	0	0	1	1	9	s_3	
-7	0	0	3	0	3	$\leftarrow$ Current z-value	

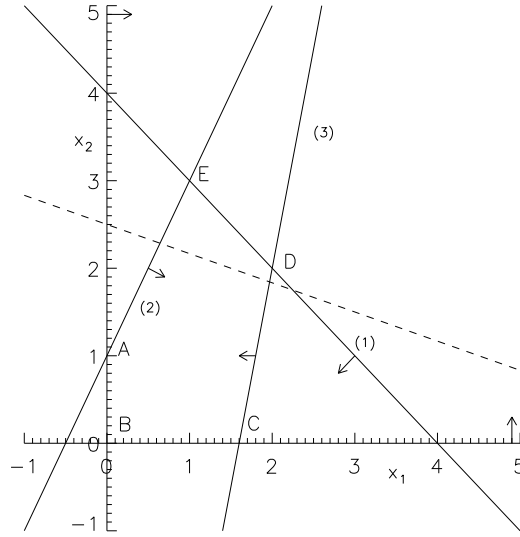
Since we have only one negative element in the last row, x_1 is the entering variable. The nonnegative ratios in the first column are $\frac{3}{3} = 1$ and $\frac{9}{3} = 3$. Hence, s_1 is the leaving variable.

x_1	x_2	s_1	s_2	s_3	b	Basic variables	
③	0	1	-1	0	3	s_1	departing
-2	1	0	1	0	1	x_2	
3	0	0	1	1	9	s_3	
-7	0	0	3	0	3	←Current z-value	
↑							
entering							

Now we divide the first (departing) row by 3 ($R_1 \rightarrow \frac{1}{3}R_1$) to make the pivot element unity. Then, with its help, we make the other elements in the first column zero by the row operations $R_2 \rightarrow R_2 + 2R_1$, $R_3 \rightarrow R_3 - 3R_1$, and $R_4 \rightarrow R_4 + 7R_1$. Thus, the simplex tableau becomes

x_1	x_2	s_1	s_2	s_3	b	Basic variables	
1	0	$\frac{1}{3}$	$-\frac{1}{3}$	0	1	x_1	
0	1	$\frac{2}{3}$	$\frac{1}{3}$	0	3	x_2	
0	0	-1	2	1	6	s_3	
0	0	$\frac{7}{3}$	$\frac{2}{3}$	0	10	$\leftarrow$ Current z-value	

Since none of the elements in the last row is negative, the solution is optimal. The optimal values are $x_1 = 1$, $x_2 = 3$, $s_1 = 0$, $s_2 = 0$, $s_3 = 6$ and $z = x_1 + 3x_2 = 1 + 3 \cdot 3 = 10$.



The feasible region for the above problem is the region inside the polygon ABCDE. The optimum value occurs at $E(1,3)$, which agrees with the solution obtain by the simplex method. Also note that our initial basic feasible solution is the vertex $B(0,0)$ with $z = 0$, this is possible only because the problem is in standard form for maximization. At the next iteration, the basic feasible solution is vertex $A(0,1)$ with $z = 3$. Note that the corner $C(\frac{8}{5}, 0)$ is also feasible but the value of $z = \frac{8}{5}$ is less than the value at A. Hence, the simplex method moves to A instead of C in the next iteration. Finally, it moves to $E(1,3)$, which is the optimal corner.

Solving minimization problem: We consider the minimization problem of the following form:

$$\text{Minimize } w = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m, \end{aligned}$$

where $x_i \geq 0$ ($i = 1, 2, \dots, n$), and $c_i \geq 0$ ($i = 1, 2, \dots, n$). The coefficients in the objective function are chosen non-negative because the dual problem of the above becomes an LPP for maximization in standard form.

To form the dual problem, we follow the following steps:

- i. We form the augmented matrix for the given system of inequalities, and add a bottom

row consisting of the coefficients of the objective function:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots & b_2 \\ & & \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & \vdots & b_m \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ c_1 & c_2 & \cdots & c_n & \vdots & 0 \end{bmatrix}$$

ii. Next we form the transpose of the matrix:

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} & \vdots & c_1 \\ a_{12} & a_{22} & \cdots & a_{m2} & \vdots & c_2 \\ & & \vdots & \vdots & & \\ a_{1n} & a_{2n} & \cdots & a_{mn} & \vdots & c_n \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ b_1 & b_2 & \cdots & b_m & \vdots & 0 \end{bmatrix}$$

iii. Form the dual maximization problem corresponding to this transposed matrix:

Maximize $z = b_1y_1 + b_2y_2 + \cdots + b_my_m$,

subject to the constraints

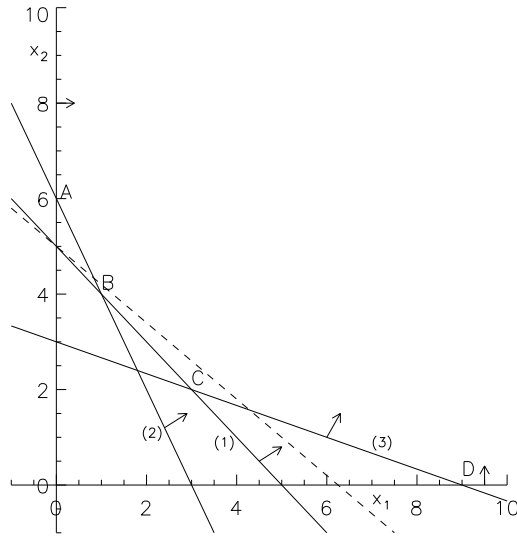
$$\begin{aligned} a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m &\leq c_1 \\ a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m &\leq c_2 \\ &\vdots \\ a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m &\leq c_n, \end{aligned}$$

where $y_i \geq 0$ for $i = 1, 2, \dots, m$. Note that $c_i \geq 0$ ($i = 1, 2, \dots, n$) also holds due to the choice of the coefficients of the original problem.

iv. Now we apply the simplex method to the dual maximization problem (which is now in the standard form). The maximum value of z will be the minimum value of w . In addition, the values of $x_1, x_2, \dots, x_n$ will appear in the bottom row of the final simplex tableau, in the columns corresponding to the slack variables.

Example: Consider the following LPP:

$$\begin{aligned} \text{Minimize} \quad & w = 4x_1 + 5x_2 \\ \text{subject to} \quad & x_1 + x_2 \geq 5, \\ & 2x_1 + x_2 \geq 6, \\ & x_1 + 3x_2 \geq 9, \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$



This problem can be solved graphically. The figure shows the feasible region bounded by the line segments AB, BC, and CD, along with the x-axis to the right of D and the y-axis above A. It is clear that the feasible region is unbounded. The objective function for $w = 25$ is shown by the dashed line. Since we minimize the objective function, the minimum occurs at $C(3, 2)$, which is the intersection point of constraints (1) and (3); and the minimum of w is 22.

The corresponding dual maximization problem is

$$\begin{aligned}
 &\text{Maximize} && z = 5y_1 + 6y_2 + 9y_3 \\
 &\text{subject to} && y_1 + 2y_2 + y_3 \leq 4, \\
 & && y_1 + y_2 + 3y_3 \leq 5, \\
 & && y_1 \geq 0, y_2 \geq 0, y_3 \geq 0.
 \end{aligned}$$

This can be solved using the corner points method as well as the simplex method. First, we introduce two slack variables $s_1 \geq 0$ and $s_2 \geq 0$:

$$\begin{aligned}
 &\text{Maximize} && z = 5y_1 + 6y_2 + 9y_3 \\
 &\text{subject to} && y_1 + 2y_2 + y_3 + s_1 = 4, \\
 & && y_1 + y_2 + 3y_3 + s_2 = 5, \\
 & && y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, s_1 \geq 0, s_2 \geq 0.
 \end{aligned}$$

Since we have two equations, we set three variables to zero and solve for the other two. Hence, the total number of corner points (all of them may not be feasible) is ${}^5C_3 = 10$.

Points	y_1	y_2	y_3	s_1	s_2	Feasible	z
1	0	0	0	4	5	Yes	0
2	0	0	5	0	-7	No	—
3	0	0	$\frac{5}{3}$	$\frac{7}{3}$	0	Yes	15
4	0	2	0	0	3	Yes	12
5	0	5	0	-6	0	No	—
6	0	$\frac{6}{5}$	$\frac{7}{5}$	0	0	Yes	$\frac{99}{5}$
7	4	0	0	0	1	Yes	20
8	5	0	0	-1	0	No	—
9	$\frac{7}{2}$	0	$\frac{1}{2}$	0	0	Yes	22✓
10	6	-1	0	0	0	No	—

It is seen that the optimal solution maximum of $z = 22$ occurs when we choose $y_2 = 0$, $s_1 = 0$ and $s_2 = 0$. If we had solved the original problem using corner points, then the surplus variables $s_1 = 0$ and $s_3 = 0$ (we have used the same notation for slack and surplus variables!). Thus

$$\begin{aligned}
&\text{Minimize} && w = 4x_1 + 5x_2 \\
&\text{subject to} && x_1 + x_2 = 5, \\
&&& 2x_1 + x_2 - s_2 = 6, \\
&&& x_1 + 3x_2 = 9, \\
&&& x_1 \geq 0, x_2 \geq 0, s_2 \geq 0.
\end{aligned}$$

Solving the first and third equations, we get $x_1 = 3, x_2 = 2$ and substituting in the second equation, we find $8 - s_2 = 6 \implies s_2 = 6$, which is feasible. Hence, we get the same optimal solution.

Now, we solve the dual maximization problem using the simplex method. The simplex tables are shown below. These are made using the procedure described in the context of the simplex method applied to the standard LPP for maximisation.

y_1	y_2	y_3	s_1	s_2	b	Basic variables	
1	2	1	1	0	4	s_1	
1	1	③	0	1	5	s_2	← departing
-5	-6	-9	0	0	0	← Current z-value	
		↑					
		entering					



y_1	y_2	y_3	s_1	s_2	b	Basic variables	
$\frac{2}{3}$	$\frac{5}{3}$	0	1	$-\frac{1}{3}$	$\frac{7}{3}$	s_1	$\leftarrow$ departing
$\frac{1}{3}$	$\frac{1}{3}$	1	0	$\frac{1}{3}$	$\frac{5}{3}$	y_3	
-2	-3	0	0	3	15	$\leftarrow$ Current z-value	
	$\uparrow$						
entering							



y_1	y_2	y_3	s_1	s_2	b	Basic variables	
$\frac{2}{5}$	1	0	$\frac{3}{5}$	$-\frac{1}{5}$	$\frac{7}{5}$	y_2	$\leftarrow$ departing
$\frac{1}{5}$	0	1	$-\frac{1}{5}$	$\frac{8}{15}$	$\frac{6}{5}$	y_3	
$-\frac{4}{5}$	0	0	$\frac{9}{5}$	$\frac{12}{5}$	$\frac{96}{5}$	$\leftarrow$ Current z-value	
$\uparrow$							
entering							



y_1	y_2	y_3	s_1	s_2	b	Basic variables
1	$\frac{5}{2}$	0	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{7}{2}$	y_1
0	$-\frac{1}{2}$	1	$\frac{1}{10}$	$\frac{19}{30}$	$\frac{1}{2}$	y_3
0	2	0	3	2	22	$\leftarrow$ Current z-value
$\uparrow$						
entering						

Thus, the optimal solution of the maximization problem is $z = 22$, which is the same as the optimal solution of the original minimization problem. Also, the entry in the bottom row under $s_1 = 3$ and $s_2 = 2$ gives the optimal solution to the original problem: $x_1 = 3$ and $x_2 = 2$.

Note that the linear programming problems used to illustrate the simplex method are of a particular form. For these, it is easy to find the starting basic feasible solution. However, the simplex method is also used for a variety of linear programming problems (e.g., equality constraints, unrestricted variables, etc.) not covered here. Similarly, the dual problem formulation has been carried out for a particular type of LPP. Dual formulation are being used for a variety for other types of linear programming problems too.