Radioactive decay, Malthusian population growth model

Example (Radioactive decay): Uranium-238 with a half-life of 4.51×10^9 years decays to thorium-234. Thorium-234 has a half-life of 0.066 years. The decay constant m for Uranium-238 is $m = \ln 2/4.51 \times 10^9 \approx 1.537 \times 10^{-10}$ year^{-1} and that of Thorium-234 is $n = \ln 2/0.066 \approx 10.50$ year^{-1}.

Let $U(t)$ and $T(t)$ be the amount of Uranium-238 and Thorium-234 in a sample at time $t \geq 0$. Let their initial amount be U_0 and T_0 respectively. Suppose that the two-element system is kept in isolation. Then, the governing equations are

$$\frac{dU}{dt} = -mU, \tag{1}$$
$$\frac{dT}{dt} = -nT + mU, \tag{2}$$

subject to the initial condition $U(0) = U_0$ and $T(0) = T_0$. Solving the above we find

$$\begin{aligned} U(t) &= U_0 \mathrm{e}^{-mt} \\ T(t) &= \frac{mU_0}{n-m}\mathrm{e}^{-mt} + \left(T_0 - \frac{mU_0}{n-m}\right)\mathrm{e}^{-nt} \end{aligned}$$

Suppose we consider a time interval $0 \leq t \leq 5000$ years. Now e^{-mt} varies from $\mathrm{e}^{-m\cdot 0} = 1$ to $\mathrm{e}^{-m\cdot 5000} \approx 0.999999$. Hence, this term is almost constant in the time interval $[0, 5000]$. Now $nT = 12 \implies T \approx 1.14$ years. Since $\mathrm{e}^{-12} \approx 6 \times 10^{-6}$, we conclude that $\mathrm{e}^{-nt} \approx 0$ for $t \geq 1$ years. Hence, for a time interval $1 \leq t \leq 5000$ years, we conclude that

$$T(t) \approx \frac{mU_0}{n-m} \approx \frac{mU_0}{n} \approx \frac{mU(t)}{n} \implies nT(t) \approx mU(t), \qquad n \gg m.$$

From (2), we find that $dT/dt \approx 0$ when $nT(t) \approx mU(t)$. Hence, we can think of a "temporary" equilibrium for $1 \leq t \leq 5000$ in which the amount of thorium generated due to uranium decay matches the amount of thorium decay. However, over a long time, both $U(t)$ and $T(t)$ tend to zero.

Example (Exponential growth population model): Let $p(t)$ be the population density at time $t \geq 0$ and $p_0 = p(t = 0)$ be the initial population. In the simplistic model, we assume that the birth rate $B(t)$ is proportional to the population density, and so is the death rate $D(t)$. Thus, $B(t) \propto p(t)$ and $D(t) \propto p(t)$. Hence, $B(t) = bp(t)$ and $D(t) = dp(t)$, where b and d are proportionality constants. The growth of the population $p(t)$ satisfies

$$\frac{dp}{dt} = B(t) - D(t) = (b - d)p = rp,$$

where $r = b - d$ is the intrinsic growth rate. This model is called Malthusian law of population growth. Since $\frac{1}{p}\frac{dp}{dt} = r$, the above equation implies that the per capita growth rate is constant. The above equation is subject to the initial condition $p(0) = p_0$. Solving, we find $p(t) = p_0 e^{rt}$. It was observed that the world's population was increasing at an average rate of 2% during 1965–1970. In the year 1965, the world population was approximately 3.34 billion. We measure time from 1965; i.e., 1965 is taken as $t = 0$. Hence, $p(t) = 3.34\mathrm{e}^{0.02t}$. One way to check the model's accuracy is to determine the time τ required for the population to double, which was observed to be approximately 35 years. From the model, we find that

$$\frac{p(t+\tau)}{p(\tau)} = 2 \implies e^{0.02\tau} = 2 \implies \tau = \frac{\ln 2}{0.02} \implies \tau \approx 34.7 \text{ years},$$

which is an excellent agreement. If we look into the future, we observe that the population in the year 2515 is $3.34\mathrm{e}^{0.02(2515-1965)} \approx 200,000$ billion. Similarly, the population in the year 2625 and 2660 would be approximately $1,800,000$ and $3,600,000$ billions, respectively. Now, Earth's total land surface area is $\approx 1,600,000$ billion square feet. Thus, by 2515, there will be around 8 square feet per person, less than 1 square feet per person by 2625, and not enough space by 2660. Consequently, the model seems unreasonable and should not be used. However, this model has offered exceptional agreement in the past. To see why the model does not always work, we observe that the assumption of a constant per capita growth rate, $\frac{1}{p}\frac{dp}{dt} = r$, holds as long as the population density is low. When the population grows large, this model does not account for the fact that individual members are now competing with one another for finite living space, food, and natural resources.

Equilibria and their stability: Consider an one-dimensional autonomous system

$$\frac{du}{dt} = f(u)$$

If $f(u) = 0$ for $u = u^*$, then u^* is an equilibrium point of the system. Note that if $u(t = 0) = u^*$, then $u(t) \equiv u^*$ is a solution of the system that stays at u^* for all time. If any solution that starts in a small neighbourhood of u^* and remains in a small neighbourhood of u^*, then we classify the equilibrium point u^* as stable; otherwise, it is unstable.

Derivate test for stability: Let u^* be an equilibrium point. Let $u^* + \epsilon\eta(t)$ is a neighbouring solution whose initial position at $t = 0$ is $u^* + \epsilon n_0$, where $0 < \epsilon \ll 1$. Now $u^* + \epsilon\eta(t)$ satisfies

$$\frac{d}{dt}(u^* + \epsilon\eta(t)) = f(u^* + \epsilon\eta(t)) \implies \epsilon\frac{d\eta}{dt} = f(u^*) + \epsilon\eta f'(u^*) + \frac{(\epsilon\eta)^2}{2!}f''(\zeta),$$

where the last term is the remainder. Since $f(u^*) = 0$ (equilibrium point), we find

$$\frac{d\eta}{dt} = f'(u^*)\eta + \frac{\epsilon\eta^2}{2!}f''(\zeta).$$

We assume that $f'(u^*) \neq 0$ and hence the last term on the RHS is negligible compared to the first term since $0 < \epsilon \ll 1$. Thus, we solve

$$\frac{d\eta}{dt} \approx f'(u^*)\eta, \qquad \eta(0) = \eta_0.$$

Solving we find

$$\eta(t) \approx \eta_0 \mathrm{e}^{f'(u^*)t}.$$

Clearly, $\eta(t) \to 0$ monotonically as t increases when $f'(u^*) < 0$, which implies that u^* is a stable equilibrium point. On the other hand, the equilibrium point is unstable if $f'(u^*) > 0$.

However, no conclusion can be drawn when $f'(u^*) = 0$. The analysis just described here is called linear stability analysis since the quadratic and higher-order terms are neglected in the Taylor series expansion.

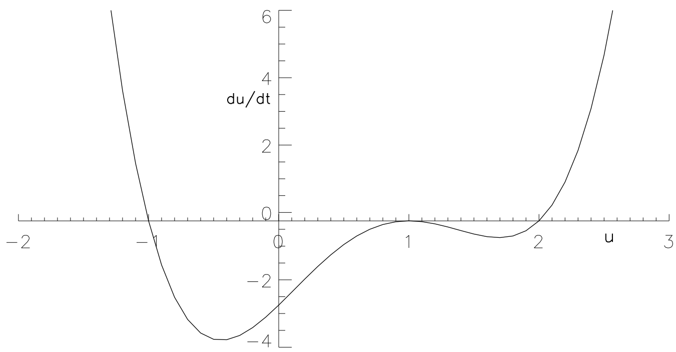
Example: Consider the autonomous system

$$\frac{du}{dt} = f(u) \equiv (u+1)(u-1)^2(u-2)$$

Solving $f(u) = 0$, we get the equilibrium points $u^* = -1, 1, 2$. Now $f'(u) = (u-1)^2(u-2) + 2(u+1)(u-1)(u-2) + (u+1)(u-1)^2$.

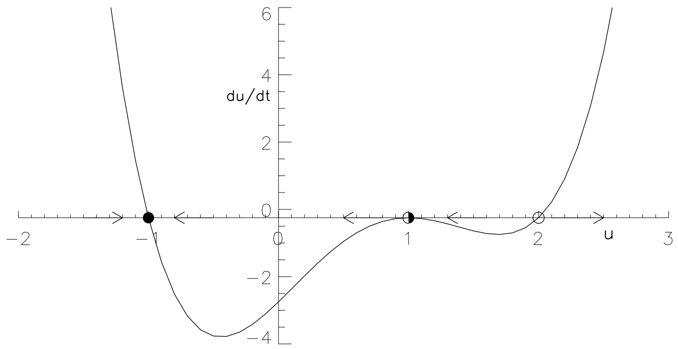
$$f'(-1) = -12 < 0, \; f'(1) = 0, \; f'(2) = 3 > 0,$$

which imply that $u^* = -1$ is stable and $u^* = 2$ is unstable. On the other hand, no conclusion can be drawn for $u^* = 1$. If we plot du/dt against u, we obtained the following diagram:

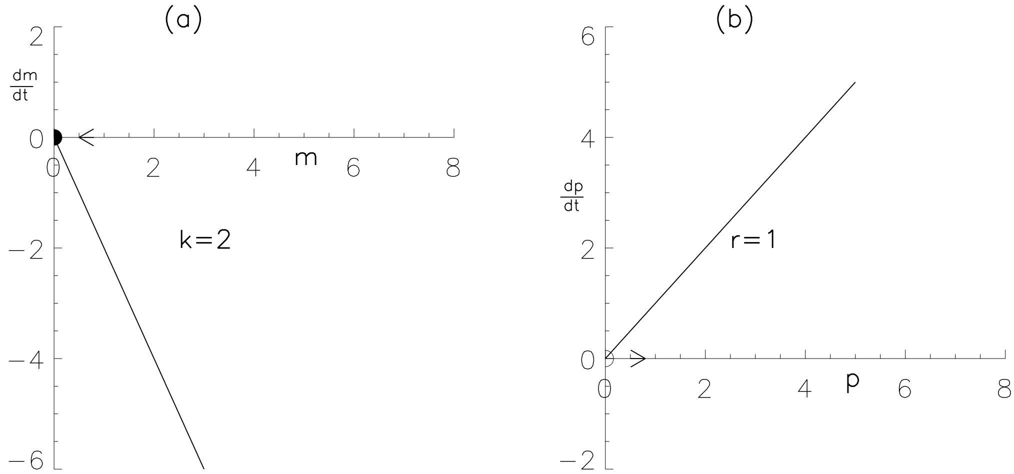


Clearly, the stable equilibrium point $u = -1$ has a negative slope, and the unstable equilibrium point $u = 2$ has a positive slope. Linear stability analysis cannot determine stability at the point $u = 1$, where the slope is horizontal.

Phase line method to test stability: Here, the drawback of the linear stability analysis is removed. However, this method is applicable only for a one-dimensional autonomous system. Here also, we plot $du/dt = f(u)$ against u. The curve $f(u)$ meets the u-axis at the equilibria. Now we draw the directions of the trajectories in the neighbourhood of an equilibrium point. If the trajectories approach the equilibrium point from both sides, then the equilibrium point is stable. On the other hand, if the trajectories move away from the equilibrium point in both directions, it is unstable. It is possible that the trajectories approach the equilibrium point from one side and move away from it on the other side. In this case, the equilibrium point is semi-stable, which necessarily implies an unstable equilibrium.



Note that if the initial condition is at an equilibrium point, then the state of the corresponding solution remains at the equilibrium point for all time. If the initial condition lies in $(2, \infty)$, the corresponding trajectories move right (see figure) since $du/dt > 0$. Similarly, the trajectories whose initial points lie in $(1, 2)$ move toward the left since $du/dt < 0$ there. Thus, $u^* = 2$ is an unstable equilibrium point represented by the open circle. Similarly, $u^* = -1$ is a stable equilibrium point represented by the solid circle. On the other hand, $u^* = 1$ is stable to the right and unstable to the left. Thus, it is semi-stable represented by a solid semicircle to the right and an open semicircle to the left. Consequently, the stability of all equilibria of the one-dimensional autonomous system can be determined.



Phase line diagram for the (a) radioactive decay model, and (b) population growth model.

The phase line diagram for the radioactive decay model $dm/dt = -km$ and population growth model $dp/dt = rp$ are shown in the figure. The trivial equilibrium point is stable for the radioactive decay and unstable for the population growth. Since the radioactive material or population are nonnegative, we construct phase line diagram in the right-half plane only.