Age-specific population growth II

Theorem: A Leslie matrix $\mathcal{L}$ has a unique positive eigenvalue λ_1 which is simple, and has a corresponding eigenvector $\boldsymbol{v}_1$ all of whose entries are positive.

Proof: If $\lambda \neq 0$, then we can write characteristic polynomial $p(\lambda) = |\lambda\mathcal{I} - \mathcal{L}|$ as

$$p(\lambda) = \lambda^n\big(1 - q(\lambda)\big), \quad \text{where } q(\lambda) = \frac{a_1}{\lambda} + \frac{a_2 b_1}{\lambda^2} + \cdots + \frac{a_n b_1 b_2 \cdots b_{n-1}}{\lambda^n}$$

Note that all $b_i > 0$ and at least one $a_i > 0$. Hence, $q(\lambda) \to \infty$ as $\lambda \to 0^+$, and $q(\lambda) \to 0$ as $\lambda \to \infty$. In addition, taking the first derivative of q with respect to λ, we find that $q(\lambda)$ is a strictly decreasing function for $\lambda > 0$.

Consequently, we can conclude that there exists a unique positive solution λ_1 of the equation $q(\lambda) = 1$. However, $q(\lambda) = 1$ implies $p(\lambda) = 0$ and hence the Leslie matrix $\mathcal{L}$ has a unique positive eigenvalue λ_1.

If $\boldsymbol{v}_1 = (w_1, w_2, \cdots, w_n)^T$ is an eigenvector corresponding to λ_1, then

$$\mathcal{L}\boldsymbol{v}_1 = \lambda_1 \boldsymbol{v}_1,$$

which implies

$$\begin{aligned} \lambda_1 w_1 &= a_1 w_1 + a_2 w_2 + \cdots + a_n w_n \\ \lambda_1 w_{i+1} &= b_i w_i, \qquad i = 1, 2, \cdots, n-1. \end{aligned}$$

From the second relation, using the fact that all the $b_i > 0$, we observe that $w_i = 0$ for $i = 2, 3, \cdots, n$, whenever $w_1 = 0$. This leads to $\boldsymbol{v}_1 = 0$, which is not possible since $\boldsymbol{v}_1$ is an eigenvector. Hence, $w_1 \neq 0$ and consequently, all the components of $\boldsymbol{v}_1$ are nonzero. Since any multiple of an eigenvector is also an eigenvector, we can take the first component w_1 to be unity. Then, using the second relation again, we find $w_2 = b_1/\lambda_1$, $w_3 = b_1 b_2/\lambda_1^2$ and so on. We also need to show that the first relation is satisfied. Substituting into the right-hand side of the first relation, we get

$$\begin{aligned} &a_1 + \frac{a_2 b_1}{\lambda_1} + \frac{a_3 b_1 b_2}{\lambda_1^2} + \cdots + \frac{a_n b_1 b_2 \cdots b_{n-1}}{\lambda_1^{n-1}} = \\ &= \lambda_1 \left(\frac{a_1}{\lambda_1} + \frac{a_2 b_1}{\lambda_1^2} + \frac{a_3 b_1 b_2}{\lambda_1^3} + \cdots + \frac{a_n b_1 b_2 \cdots b_{n-1}}{\lambda_1^n} \right) = \lambda_1 q(\lambda_1) = \lambda_1 = \text{left-hand side} \end{aligned}$$

Thus,

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ \frac{b_1}{\lambda_1} \\ \frac{b_1 b_2}{\lambda_1^2} \\ \vdots \\ \frac{b_1 b_2 \cdots b_{n-1}}{\lambda_1^{n-1}} \end{pmatrix}$$

© S Ghorai

Thus, a Leslie matrix $\mathcal{L}$ has a unique positive eigenvalue λ_1 which is simple, and has a corresponding eigenvector $\boldsymbol{v}_1$ all of whose entries are positive.

Dominant eigenvalue: If two successive entries in the first row of the Leslie matrix $\mathcal{L}$ (say a_i and a_{i+1}) are both positive, then for any eigenvalue $\lambda \neq \lambda_1$ of $\mathcal{L}$, we have $|\lambda| < \lambda_1$. Thus, for two successive fertile classes, the positive eigenvalue λ_1 is dominant.

Proof: Let λ be any nonzero eigenvalue and λ_1 be the unique positive eigenvalue. We need to show that $|\lambda| < \lambda_1$ when two successive fertile classes exist.

First, we show that $|\lambda| \leq \lambda_1$. Let $\lambda = re^{i\theta}$ ($r \neq 0$). If we denote a_1 by f_1, a_2b_1 by f_2 and so on, then $q(\lambda) = 1$ is equivalent to

$$\frac{f_1}{r}e^{-i\theta} + \frac{f_2}{r^2}e^{-i2\theta} + \cdots + \frac{f_n}{r^n}e^{-in\theta} = 1$$

Taking modulus on both sides and using the triangle inequality, we get

$$\frac{f_1}{r} + \frac{f_2}{r^2} + \cdots + \frac{f_n}{r^n} \geq 1 \implies q(r) \geq q(\lambda_1)$$

Since $q(\lambda)$ is strictly decreasing for $\lambda > 0$, $q(r) \geq q(\lambda_1)$ implies $r \leq \lambda_1 \implies |\lambda| \leq \lambda_1$.

Next, we show the dominance of λ_1 for two successive fertile classes. Suppose, on the contrary, there exists $\lambda \neq \lambda_1$ such that $|\lambda| = r = \lambda_1$. Then we have

$$q(r) = q(\lambda_1) \implies \frac{f_1}{r} + \frac{f_2}{r^2} + \cdots + \frac{f_n}{r^n} = 1,$$

which implies

$$\left|\frac{f_1}{r}e^{-i\theta}\right| + \left|\frac{f_2}{r^2}e^{-i2\theta}\right| + \cdots + \left|\frac{f_n}{r^n}e^{-in\theta}\right| = \frac{f_1}{r}e^{-i\theta} + \frac{f_2}{r^2}e^{-i2\theta} + \cdots + \frac{f_n}{r^n}e^{-in\theta} = 1$$

Note that in each of the above sums, at least two consecutive terms are nonzero.

$\Big($Note that for two nonzero complex numbers z_1 and z_2, suppose that $z_1 + z_2 = |z_1| + |z_2|$. Then, this implies

$$|z_1 + z_2| = |z_1| + |z_2| \implies \arg(z_1) = \arg(z_2) \implies z_1 = r_1e^{i\psi},\ z_2 = r_2e^{i\psi}$$

Hence, using $z_1 + z_2 = |z_1| + |z_2|$, we find

$$e^{i\psi}(r_1 + r_2) = r_1 + r_2 \implies e^{i\psi} = 1 \implies \psi = 0(\text{modulo}2\pi)$$

Hence, $z_1 > 0$ and $z_2 > 0$. This results can be generalised.$\Big)$

Here, suppose $a_j > 0$ and $a_{j+1} > 0$. Then

$$\frac{f_j}{\lambda^j} > 0 \quad \text{and} \quad \frac{f_{j+1}}{\lambda^{j+1}} > 0$$

Since $f_j > 0$ and $f_{j+1} > 0$, dividing the relation we find $\lambda > 0$. Note that $\frac{f_j}{\lambda^j} > 0$ does not imply $\lambda > 0$, for example, take $\lambda = e^{-i\pi}$ and $j = 2$.

Since $\lambda > 0$, we have $\lambda_1 = r = \lambda$. This contradiction shows that the unique positive eigenvalue is dominant.

Limiting distribution: Suppose there are two consecutive fertile classes and the Leslie matrix $\mathcal{L}$ is diagonalisable. Let λ_1 be the dominant eigenvalue and $\boldsymbol{v}_1$ (mentioned earlier) be the corresponding eigenvector. We denote the other eigenvalues and eigenvectors by $\lambda_2, \lambda_3, \cdots, \lambda_n$ and $\boldsymbol{v}_2, \boldsymbol{v}_3, \cdots, \boldsymbol{v}_n$, respectively. Then

$$\mathcal{L}^k = \mathcal{P}\text{diag}(\lambda_1^k, \lambda_2^k, \cdots, \lambda_n^k)\mathcal{P}^{-1}, \quad \text{where} \quad \mathcal{P} = \begin{pmatrix} \boldsymbol{v}_1 & \vdots & \boldsymbol{v}_2 & \vdots & \boldsymbol{v}_3 & \vdots & \cdots & \vdots & \boldsymbol{v}_n \end{pmatrix}$$

Consequently, the (female) population distribution $\boldsymbol{x}^{(k)}$ in the k-th time period and the initial (female) population $\boldsymbol{x}^{(0)}$ are related by

$$\boldsymbol{x}^{(k)} = \mathcal{L}^k \boldsymbol{x}_0 = \mathcal{P} \begin{pmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n^k \end{pmatrix} \mathcal{P}^{-1}\boldsymbol{x}_0, \qquad k = 1, 2, \cdots$$

Dividing both sides by λ_1^k, we find

$$\frac{1}{\lambda_1^k}\boldsymbol{x}^{(k)} = \mathcal{P} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \left(\frac{\lambda_2}{\lambda_1}\right)^k & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \left(\frac{\lambda_n}{\lambda_1}\right)^k \end{pmatrix} \mathcal{P}^{-1}\boldsymbol{x}_0, \qquad k = 1, 2, \cdots$$

Since λ_1 is the dominant eigenvalues, $\left|\frac{\lambda_i}{\lambda_1}\right| < 1$ for $i = 2, 3, \cdots, n$. Hence,

$$\left(\frac{\lambda_i}{\lambda_1}\right)^k \to 0 \quad \text{as } k \to \infty \quad \text{for } i = 2, 3, \cdots, n.$$

Therefore,

$$\lim_{k\to\infty} \left\{ \frac{1}{\lambda_1^k}\boldsymbol{x}^{(k)} \right\} = \mathcal{P} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \mathcal{P}^{-1}\boldsymbol{x}_0, \qquad k = 1, 2, \cdots$$

Let us denote the first entry of the column vector $\mathcal{P}^{-1}\boldsymbol{x}_0$ by c. *It can be proved that c is positive.* Now,

$$\lim_{k\to\infty} \left\{ \frac{1}{\lambda_1^k}\boldsymbol{x}^{(k)} \right\} = P \begin{pmatrix} c \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{v}_1 & \vdots & \boldsymbol{v}_2 & \vdots & \boldsymbol{v}_3 & \vdots & \cdots & \vdots & \boldsymbol{v}_n \end{pmatrix} \begin{pmatrix} c \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = c\boldsymbol{v}_1$$

Hence, we obtain the estimate

$$\boldsymbol{x}^{(k)} \approx c\lambda_1^k \boldsymbol{v}_1 \qquad \text{for large values of } k.$$

This implies that the proportion of the population in each age class remains constant in the long run. Also, for large k, we observe that $\boldsymbol{x}^{(k-1)} \approx c\lambda_1^{k-1}\boldsymbol{v}_1$ and hence $\boldsymbol{x}^{(k)} \approx \lambda_1 \boldsymbol{x}^{(k-1)}$. Therefore, the population grows by a factor of λ_1 in every time period in the long run. Further, in the long run, the proportion of the female population in each age class can be determined from the dominant eigenvector.

Example: Consider the following Leslie matrix for a certain female population:

$$\mathcal{L} = \begin{pmatrix} 0 & 4 & 3 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix}$$

The characteristic polynomial is

$$p(\lambda) = |\lambda I - \mathcal{L}| = \det \begin{pmatrix} \lambda & -4 & -3 \\ -\frac{1}{2} & \lambda & 0 \\ 0 & -\frac{1}{4} & \lambda \end{pmatrix} = \lambda(\lambda^2 - 2) + \frac{1}{4}\left(\frac{-3}{2}\right) = \lambda^3 - 2\lambda - \frac{3}{8}$$

We see that $\lambda_1 = \frac{3}{2}$ and

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ \frac{b_1}{\lambda_1} \\ \frac{b_1 b_2}{\lambda_1^2} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{3} \\ \frac{1}{18} \end{pmatrix}$$

Hence,

$$\boldsymbol{x}^{(k)} \approx c\left(\frac{3}{2}\right)^k \begin{pmatrix} 1 \\ \frac{1}{3} \\ \frac{1}{18} \end{pmatrix} \quad \text{and} \quad \boldsymbol{x}^{(k)} \approx \frac{3}{2}\boldsymbol{x}^{(k-1)} \qquad \text{for large } k.$$

Thus, in the long run, the population in each age class grows by a factor of $\frac{3}{2}$ per time period, and the proportion of population in each age class is in the ratio $1 : \frac{1}{3} : \frac{1}{18}$ or $18 : 6 : 1$. Also, since $\lambda_1 > 1$, the population survives in the long run.

Example: Consider the following Leslie matrix for a certain female population:

$$\mathcal{L} = \begin{pmatrix} 0 & 2 & 1 \\ \frac{2}{5} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix}$$

The characteristic polynomial in this case is

$$p(\lambda) = |\lambda I - \mathcal{L}| = \det \begin{pmatrix} \lambda & -2 & -1 \\ -\frac{2}{5} & \lambda & 0 \\ 0 & -\frac{1}{4} & \lambda \end{pmatrix} = \lambda\left(\lambda^2 - \frac{4}{5}\right) + \frac{1}{4}\left(\frac{-2}{5}\right) = \lambda^3 - \frac{4}{5}\lambda - \frac{1}{10}$$

We observe that $p(0) = -\frac{2}{10} < 0$ and $p(1) = \frac{1}{10} > 0$. Thus, the dominant eigenvalue λ_1 is such that $0 < \lambda_1 < 1$. In the long run,

$$\boldsymbol{x}^{(k)} \approx c\,(\lambda_1)^k\,\boldsymbol{v}_1 \implies \boldsymbol{x}^{(k)} \approx \boldsymbol{0}.$$

Hence, the population becomes extinct in the long run.