Constant harvesting model

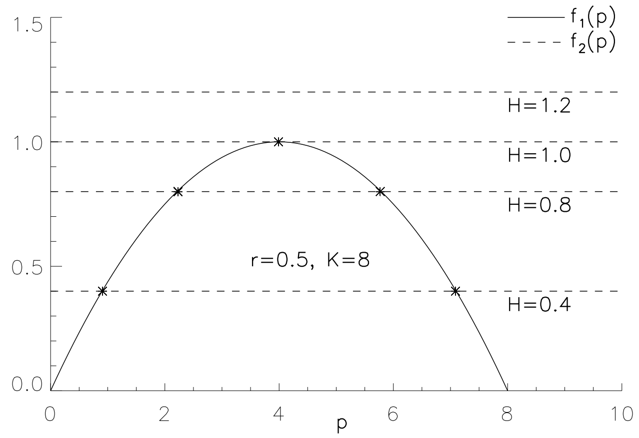
Here, a given population is being harvested at a constant rate of H per unit time. In the absence of harvesting, the population with size p grows logistically with intrinsic growth rate r and carrying capacity K. Thus, the constant harvesting model is given by

$$\frac{dp}{dt} = rp\left(1 - \frac{p}{K}\right) - H, \qquad p(0) = p_0 > 0.$$

We write

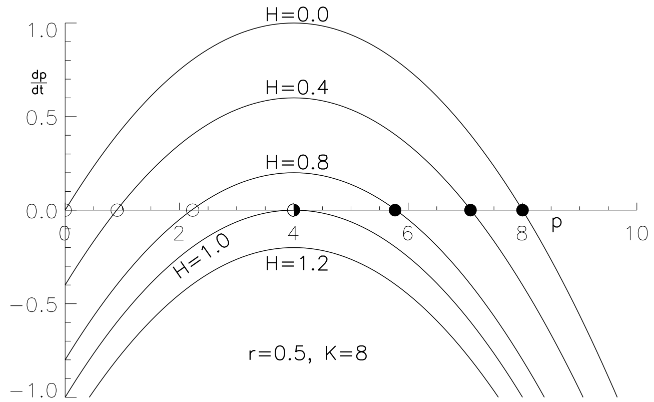
$$\frac{dp}{dt} = f(p) \equiv f_1(p) - f_2(p), \qquad \text{where} \quad f_1(p) = rp\left(1 - \frac{p}{K}\right), \ f_2(p) = H.$$

Note that $f_1(p)$ is independent of H. The equilibria are obtained by solving $f(p) = 0$, i.e., find the values of p at which $f_1(p) = f_2(p)$.



We know that $f_1(p)$ is an inverted parabola with vertex at $\left(\frac{K}{2}, \frac{rK}{4}\right)$. Hence, maximum value of $f_1(p)$ is $\frac{rK}{4}$. Since H is a constant, $f_2(p)$ is a horizontal line parallel to the p axis. When $0 < H < \frac{rK}{4}$, then $f_1(p)$ and $f_2(p)$ intersects at two points which are symmetric about $\frac{K}{2}$. When $H = \frac{rK}{4}$, then they touch at one point, leading to a single equilibrium point. However, $f_1(p)$ and $f_2(p)$ do not intersect and thus no equilibrium point exists when $H > \frac{rk}{4}$. The figure above shows the number of equilibria (marked by asterisks) as the harvesting rate H varies, with $r = 0.5$ and $K = 8$.

To draw the phase line diagram, we plot $\frac{dp}{dt} = f(p)$ against p. Since $f_2(p) = H$ is constant, the curve $f(p)$ is obtained by shifting the $f_1(p)$ curve by a distance H vertically downwards.



When $H = 0$, then the equilibria are $p^* = 0$ and $p^* = K$, which are the same as those of the logistic model. As H increases from zero, the left and right equilibria shift towards $\frac{K}{2}$. These two equilibria are solution of $f(p) = 0$:

$$rp\left(1 - \frac{p}{K}\right) - H = 0 \implies p^*_\pm = \frac{K}{2}\left(1 \pm \sqrt{1 - \frac{4H}{Kr}}\right)$$

The two equilibria are distinct when $H < \frac{rK}{4}$ and they coincide when $H = \frac{rK}{4}$. The real roots become complex conjugate, and thus no equilibria exist when $H > \frac{rK}{4}$. The larger real root is denoted by p^*_+, and the smaller is denoted by p^*_-. When $H = \frac{rK}{4}$, then $p^*_+ = p^*_- = \frac{K}{2}$. From the phase line diagram, we see that p^*_+ is stable and p^*_- is unstable when $H < \frac{rK}{4}$. When $H = \frac{rK}{4}$, then the equilibrium point is semi-stable. The phase line diagram also shows that $\frac{dp}{dt} < 0$ and hence the population goes to extinction for $H > \frac{rK}{4}$.

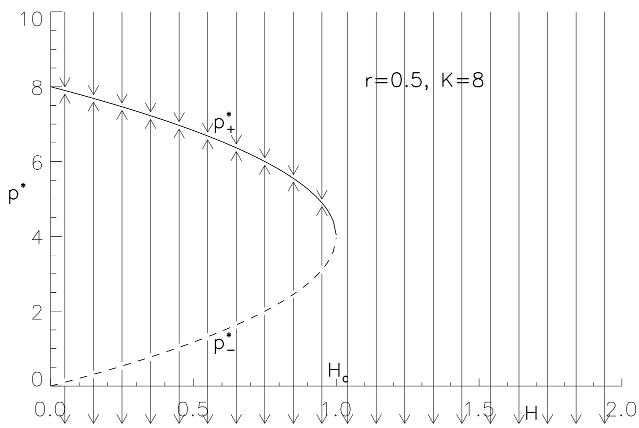
Note that we can write the growth rate as follows

$$\frac{dp}{dt} = f(p) \equiv \left(\frac{rK}{4} - H\right) - \frac{r}{K}\left(p - \frac{K}{2}\right)^2$$

This shows that $\frac{dp}{dt} < 0$ for $H > \frac{rK}{4}$ leading to population extinction. We also have

$$f'(p) = -\frac{2r}{K}\left(p - \frac{K}{2}\right) \implies f'(p^*_\pm) = -\frac{2r}{K}\left(p^*_\pm - \frac{K}{2}\right)$$

For $H < \frac{rK}{4}$, we conclude that $f'(p^*_+) < 0$ and $f'(p^*_-) > 0$ since $p^*_+ > \frac{K}{2}$ and $p^*_- < \frac{K}{2}$. Thus, p^*_+ is stable and p^*_- is unstable for $H < \frac{rK}{4}$. Also, if the initial population p_0 exceeds p^*_-, then the system approaches the stable equilibrium p^*_+ as $t \to \infty$. On the other hand, if the initial population p_0 is positive but less than p^*_-, then the population becomes extinct. This model differs from the logistic model in this respect: for the logistic model, any positive initial population approaches the carrying capacity as time increases.



When $H = 0$, then the equilibria are 0 and K. For $0 < H < \frac{rK}{4}$, the equilibria p^*_- and p^*_+ are unstable and stable respectively. In the figure above, we plot these equilibria as a function of H. The stable branch is represented by a solid line, and the unstable branch is represented by a dashed line. As H increases from zero, p^*_- increases and p^*_+ decreases and both become equal to $\frac{K}{2}$ at $H = H_c = \frac{rK}{4}$. For $H > H_c$, no equilibrium point exists. Thus, the system dynamics consists of one stable and one unstable equilibria for $H < H_c$, and both of which vanish when H crosses the threshold value H_c. This is an example of saddle-node bifurcation. A bifurcation occurs when a small, smooth change in the parameter value(s) of a system causes a sudden qualitative or topological change in its behaviour. The parameter H is called the bifurcation parameter, and H_c is the saddle-node bifurcation threshold. In the figure above, the arrows indicate the direction of the trajectories. For example, for $0 \leq H < H_c$:

i. The solution $p(t) \to -\infty$ as $t \to \infty$ for $0 < p_0 < p^*_-$ (down arrows below the parabola).

ii. The solution $p(t) \to p^*_+$ as $t \to \infty$ for $p^*_- < p_0 < p^*+$ (up arrows inside the parabola).

iii. The solution $p(t) \to p^*_+$ as $t \to \infty$ for $p_0 > p^*+$ (down arrows above the parabola).

If $H > H_c$, then the solution $x(t) \to -\infty$ as $t \to \infty$ (down arrows to the right of H_c) for any initial population.

Maximum sustainable yield: Here H is the harvested amount per unit time. We see that H can be increased to $\frac{rK}{4}$ while keeping the population in a stable equilibrium state. Thus, $H = \frac{rK}{4}$ is called the maximum sustainable yield (MSY). Going beyond MSY leads to population extinction. Thus, harvesting close to MSY can be dangerous since a little perturbation can lead to population extinction.