Constant effort harvesting model

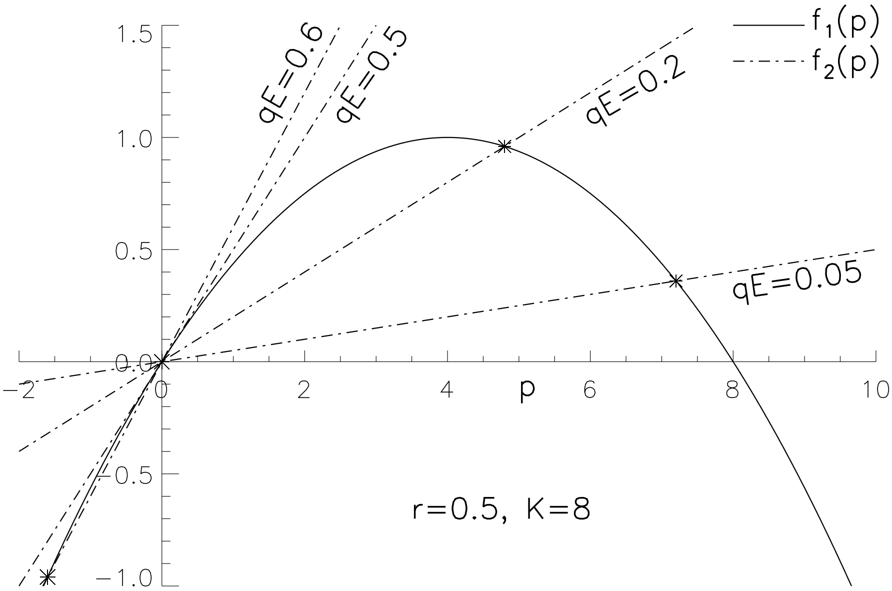
Here, we assume that the harvesting rate H is proportional to the population p. It is convenient to write the proportionality constant as the product of two components: a catchability quotient, denoted by q, and a measure of harvesting effort, which we denote by E. Hence, the harvesting rate is $H = qEp$. In the absence of harvesting, the population with size p grows logistically with intrinsic growth rate r and carrying capacity K. Thus, the constant effort harvesting model is given by

$$\frac{dp}{dt} = rp\left(1 - \frac{p}{K}\right) - qEp, \qquad p(0) = p_0 > 0.$$

We write

$$\frac{dp}{dt} = f(p) \equiv f_1(p) - f_2(p), \qquad \text{where} \quad f_1(p) = rp\left(1 - \frac{p}{K}\right), \; f_2(p) = qEp.$$

Note that $f_1(p)$ is independent of qE. The equilibria are obtained by solving $f(p) = 0$, i.e., find the values of p at which $f_1(p) = f_2(p)$.



We know that $f_1(p)$ is an inverted parabola with vertex at $\left(\frac{K}{2}, \frac{rK}{4}\right)$. Hence, maximum value of $f_1(p)$ is $\frac{rK}{4}$. On the other hand, $f_2(p)$ is a straight line with slope qE. If $0 < qE < r$, then $f_1(p)$ and $f_2(p)$ intersects at two points leading to two equilibria. If $qE < r$, then one equilibrium point is always $p_t^* = 0$, and the other feasible equilibrium point $p_e^* > 0$ shifts toward $p_t^* = 0$ as qE increases from zero. When $qE = r$, then p_e^* coincides with $p_t^* = 0$, leading to a single trivial equilibrium point $p_t^* = 0$. For $qE > r$, one equilibrium point is again $p_t^* = 0$ and the other equilibrium point p_e^* becomes negative, which is unfeasible. The figure above shows the number of equilibria (marked by asterisks) as the harvesting rate qE varies, with $r = 0.5$ and $K = 8$.

To draw the phase line diagram, we plot $\frac{dp}{dt} = f(p)$ against p. Note that when $qE = r$, then

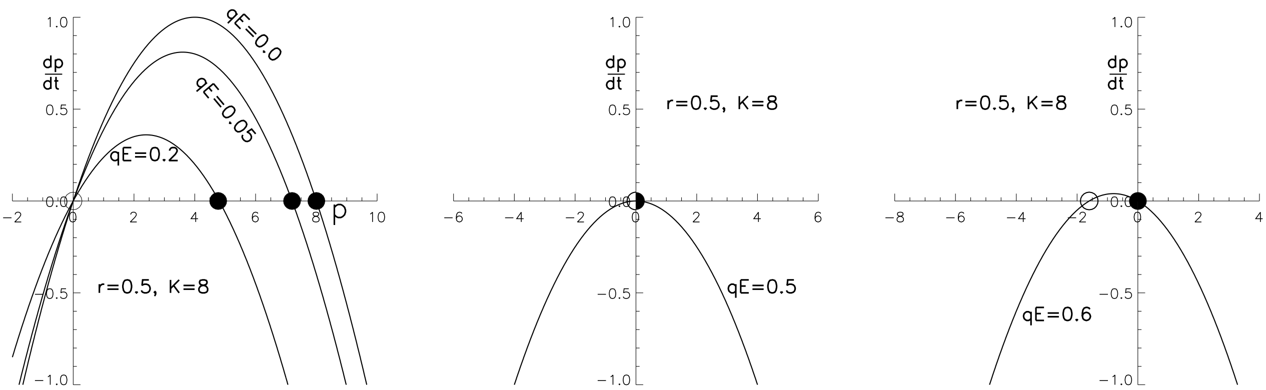
$$\frac{dp}{dt} = -\frac{r}{K}p^2,$$

which is an inverted parabola with the vertex at the origin.

For $qE \neq r$, we write

$$\frac{dp}{dt} = rp\left(1 - \frac{p}{K}\right) - qEp = r\left(1 - \frac{qE}{r}\right)p\left(1 - \frac{p}{K\left(1 - \frac{qE}{r}\right)}\right).$$

Comparing with logistic equation, for $qE \neq r$, the above is a inverted parabola with vertex at $\left(\frac{K}{2}\left(1 - \frac{qE}{r}\right), \frac{rK}{4}\left(1 - \frac{qE}{r}\right)^2\right)$. Thus, the vertex of the parabola lies in the first quadrant for $qE < r$ and it lies in the second quadrant for $qE > r$.

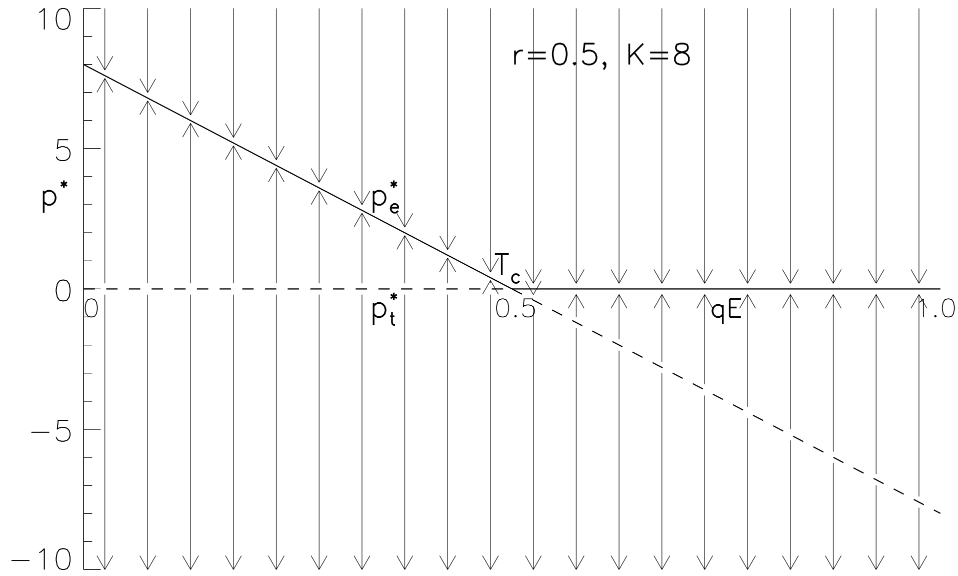


When $qE = 0$, then the equilibria are $p_t^* = 0$ and $p_e^* = K$, which are the same as those of the logistic model. As qE increases from zero with $qE < r$, the right equilibrium p_e^* shifts toward the left equilibrium point $p_t^* = 0$. These two equilibria are solution of $f(p) = 0$:

$$rp\left(1 - \frac{p}{K}\right) - qEp = 0 \implies p_t^* = 0, \quad p_e^* = K\left(1 - \frac{qE}{r}\right).$$

From the phase line diagram (see the left panel), we conclude that p_t^* is unstable and p_e^* is stable when $qE < r$. When $qE = r$, then the only equilibrium point $p_t^* = 0$ is semi-stable (see middle panel). On the other hand, for $qE > r$, the trivial equilibrium point $p_t^* = 0$ is stable, and the unfeasible nontrivial equilibrium point p_e^* becomes unstable (see the right panel). If the initial population p_0 is positive and $qE < r$, then the population approaches the right equilibrium point p_e^* as time increases. In this way, it differs from the constant harvesting model, in which the population goes extinct when $0 < p_0 < p_-^*$.

When $qE = 0$, then the equilibria are 0 and K. For $0 < qE < r$, the equilibria p_t^* and p_e^* are unstable and stable, respectively. On the other hand, the equilibria p_t^* and p_e^* respectively are stable and unstable for $qE > r$. In the figure, we plot these equilibria as a function of qE. The stable branch is represented by a solid line, and the unstable branch is represented by a dashed line. As qE increases from zero, the stable branch p_e^* decreases and the unstable branch p_t^* remains constant. They become equal to 0 at $qE = T_c = r$. At $qE = T_c$, the branches interchange their stability and the stable branch p_e^* becomes unstable, and the unstable branch p_t^* becomes stable. Thus, the system dynamics change when qE crosses the threshold value T_c. This is an example of a transcritical bifurcation, where qE is the bifurcation parameter and T_c is the transcritical bifurcation threshold.



In the figure above, the arrows indicate the direction of the trajectories. For example, when $0 \leq qE < T_c$:

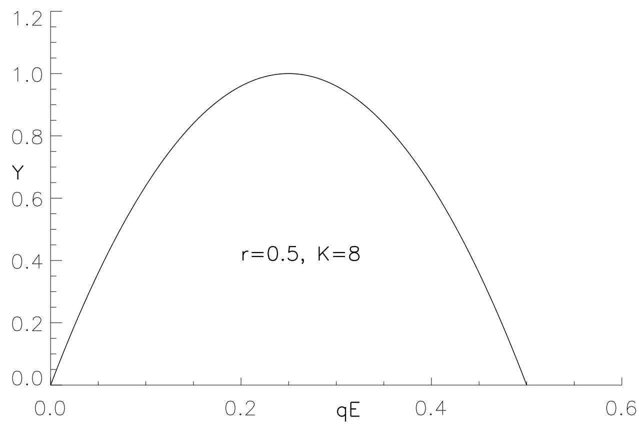
i. The solution $p(t) \to -\infty$ as $t \to \infty$ for $p_0 < 0$ (down arrows below the qE-axis).

ii. The solution $p(t) \to p_e^*$ as $t \to \infty$ for $p_0 > 0$ (up and down arrows below and above the p_e^* branch).

Similarly, other arrows can be explained.

Maximum sustainable yield: Now we calculate how much is harvested in the stable equilibrium state, which is also called sustainable yield Y:

$$Y = qEp_e^* = KqE\left(1 - \frac{qE}{r}\right).$$



The yield-effort curve is shown here. It shows that as the effort increases from small values, the sustainable yield Y increases. By the derivative test or otherwise, we observe that the yield attains its maximum value $\frac{rK}{4}$ at $qE = \frac{r}{2}$ and then it decreases to zero. Thus, maximum sustainable yield (MSY) is $\frac{rK}{4}$ which happens when the effort is $\frac{r}{2}$. Even if we exceed the effort level for MSY, the population remains in a stable equilibrium state. However, in the constant harvesting model, the population become extinct when we exceed the MSY harvesting rate.