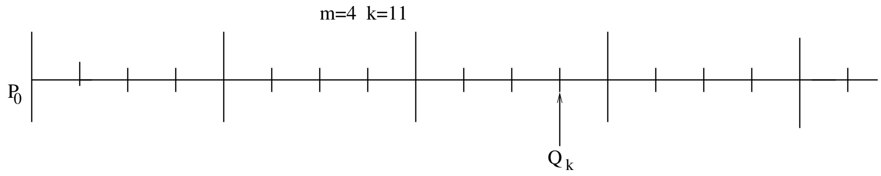
Mathematical finance

The most basic application of mathematical modelling in finance is calculating interest income using different compounding methods. Let r be the annual interest rate. For example, $r = 0.05$ if the annual interest paid by a bank is 5%. Suppose each year is divided into m sub-intervals, and interest is paid at the end of each sub-interval. Then the interest in each sub-interval is $\frac{r}{m}$. Let P_0 be the amount invested initially and $Q(k)$ be the total amount at the end of k-th interval. Note that the interval for k is counted from the beginning (see figure). Further, suppose that $P(n)$ is the total amount at the end of n years.



i. Annual compounding: Here $m = 1$ and $P(1) = Q(1) = P_0(1 + r)$. Similarly, $P(2) = Q(2) = P(1)(1 + r) = P_0(1 + r)^2$. Thus

$$P(n) = Q(n) = P_0(1 + r)^n$$

ii. Semi-annual compounding: Here $m = 2$ and $Q(1) = P_0(1 + \frac{r}{2})$ and $P(1) = Q(2) = P_0(1 + \frac{r}{2})^2$. Similarly, $P(2) = P_0(1 + \frac{r}{2})^4$. Thus

$$P(n) = Q(2n) = P_0\left(1 + \frac{r}{2}\right)^{2n}$$

Note that n, the number of years, may not be a whole number. For example, we can think of the total amount at the end of $n = 1.5$ years. Then,

$$P(1.5) = Q(3) = P_0\left(1 + \frac{r}{2}\right)^3$$

iii. Quarterly compounding: Here $m = 4$ and $Q(1) = P_0(1+\frac{r}{4})$ and $P(1) = Q(4) = P_0(1+\frac{r}{4})^4$. Similarly, $P(2) = P_0(1 + \frac{r}{4})^8$. Thus

$$P(n) = Q(4n) = P_0\left(1 + \frac{r}{4}\right)^{4n}$$

iv. Monthly compounding: Here $m = 12$ and $Q(1) = P_0(1 + \frac{r}{12})$ and $P(1) = Q(12) = P_0(1 + \frac{r}{12})^{12}$. Similarly, $P(2) = P_0(1 + \frac{r}{12})^{24}$. Thus

$$P(n) = Q(12n) = P_0\left(1 + \frac{r}{12}\right)^{12n}$$

v. Daily compounding: Here $m = 365$ and $Q(1) = P_0(1 + \frac{r}{365})$ and $P(1) = Q(365) = P_0(1 + \frac{r}{365})^{365}$. Similarly, $P(2) = P_0(1 + \frac{r}{365})^{730}$. Thus

$$P(n) = Q(365n) = P_0 \left(1 + \frac{r}{365}\right)^{365n}$$

vi. Continuous compounding: For m sub-interval in a year, we know that $Q(1) = P_0(1 + \frac{r}{m})$ and hence

$$P(n) = Q(mn) = P_0 \left(1 + \frac{r}{m}\right)^{mn}$$

We make $m \to \infty$ for continuous compounding. Hence

$$P(n) = P_0 \lim_{m\to\infty} \left(1 + \frac{r}{m}\right)^{mn} = P_0 \left(\lim_{m\to\infty} \left(1 + \frac{r}{m}\right)^{m}\right)^{n} = \mathrm{e}^{rn}$$

Thus, for continuous compounding, the amount after t years is

$$P(t) = P_0 \mathrm{e}^{rt}$$

If we subtract P_0 from the total amount $P(n)$, then we obtain the total interest due to corresponding compounding. Table below shows the total interest due to principal sum of Rs. 1 lakh deposited in a bank for 10 years assuming 5% annual interest rate.

Compounding	Total interest accrued (Rs.)
Annualy	62889.46
Semi-annualy	63861.64
Quarterly	64361.95
Monthly	64700.95
Daily	64866.48
Continuous	64872.13

This table shows that as the number of intervals per year, in which interest is paid, increases, the total accrued interest also increases.

Continuous compounding: We have seen that for an annual interest of r, the total amount accrued in t years due to continuous compounding is $P_0 e^{rt}$, where P_0 is the principal amount. This was derived by taking the number of intervals per year to be very large. Here, we show that the same result can be derived as a solution of a differential equation. Let m the intervals per year with length $\Delta t = \frac{1}{m}$. Now

$$P(t + \Delta t) = P(t) \left(1 + \frac{r}{m}\right) \implies P(t + \Delta t) - P(t) = rP(t)\Delta t,$$

where we have used the relation $m = \frac{1}{\Delta t}$. Dividing both side by Δt and taking $\Delta t \to 0$ (equivalent to $m \to \infty$), we find

$$\lim_{\Delta t\to 0} \frac{P(t + \Delta t) - P(t)}{\Delta t} = rP(t) \implies \frac{dP(t)}{dt} = rP(t),$$

which is subject to initial condition $P(0) = P_0$. Its solution is $P = P_0 e^{rt}$, which is the same obtained earlier.

Mortgage: A mortgage is a loan from a bank or lender used to buy a house or other real estate. The property serves as collateral or security. If you fail to repay the loan, the lender can take the property. Loosely speaking, with a mortgage, you borrow money to buy a home. You pay the lender back over time, like 10 or 20 years. Each payment has a part for the loan cost and a part for interest.

Let P_0 be the initial outstanding amount, and r be the annual interest rate. Let the mortgage duration be n years. The fixed payment per instalment is x, and the instalment can be annual, semi-annual, quarterly, etc. Let $P(i)$ be the debt remaining after i-th payment. Clearly, $P(0) = P_0$.

First, consider the annual payment. In this case

$$P(1) = P(0)(1+r) - x = P_0(1+r) - x, \quad P(2) = P(1)(1+r) - x = P_0(1+r)^2 - [1+(1+r)]x$$

By induction, we get

$$P(i) = P_0(1+r)^i - x\sum_{k=0}^{i-1}(1+r)^k = P_0(1+r)^i - \frac{x}{r}[(1+r)^i - 1]$$

We need to find n such that $P(n) = 0$, which implies

$$P_0(1+r)^n - \frac{x}{r}[(1+r)^n - 1] = 0 \implies x = \frac{P_0 r(1+r)^n}{(1+r)^n - 1}$$

On the other hand, if we consider payment m times per year, then $m = 1$ for annual payments, $m = 4$ for quarterly payments, $m = 12$ for monthly payments, etc. Now the number of payments in n years is mn with interest $\frac{r}{m}$ in each interval, and we want $P(mn) = 0$. Thus, we find

$$P(mn) = P_0\left(1+\frac{r}{m}\right)^{mn} - \frac{xm}{r}\left[\left(1+\frac{r}{m}\right)^{mn} - 1\right] = 0 \implies x = \frac{P_0\frac{r}{m}\left(1+\frac{r}{m}\right)^{mn}}{\left(1+\frac{r}{m}\right)^{mn} - 1}$$

Often, the number of instalments is fixed, say m, and the interest per time interval is fixed too, say R. Note that the number of instalments, m, is not per year but for the entire duration. Also, R is the interest rate per interval; it is unrelated to the annual interest rate. In this case

$$P(m) = P_0(1+R)^m - \frac{x}{R}[(1+R)^m - 1] = 0 \implies x = \frac{P_0 R(1+R)^m}{(1+R)^m - 1}$$

Present value analysis: Suppose the annual interest rate is 8% compounded yearly and you buy 10 gm silver bar for Rs. 1000/-. After one year, you sell the silver bar for Rs. 1100/-. Ignoring inflation, can we say that the silver bar's true worth at the time of purchase is Rs. 1100/-? The answer is clearly no, since Rs. 1100/- now would be Rs. $1100 \times 1.08 =$ Rs. 1188/- after one year. It is also not Rs. 1000/- since Rs. 1000/- now would be Rs. $1000 \times 1.08 =$ Rs. 1080/- after one year. Hence, the true worth of the silver bar now, called present value (PV), must lie between Rs. 1000/- and Rs. 1100/-.

If W is the present value and $r = 0.08$ is the annual interest rate, then

$$W(1+r) = 1100 \implies W = \frac{1100}{1.08} \approx 1019$$

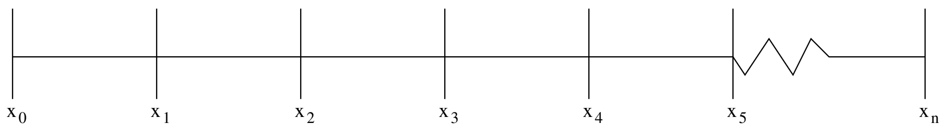
Hence, the true worth of the silver bar today is approximately Rs. 1019/-. Similarly, if the interest is compounded continuously, then the present value W satisfies

$$W\mathrm{e}^{r} = 1100 \implies W = 1100 \times \mathrm{e}^{-r} \implies W \approx 1015$$

Since the present value (PV) is higher than the purchase value, you gain by selling after one year.

The previous discussion shows that future money needs to be converted to present value for comparison. Alternatively, we may convert the present value to future value (FV) for comparison too. The future value of the purchase price after one year is Rs. 1080/-, which is less than the selling price of Rs. 1100/-. Hence, we draw the same conclusion again.

Present value/Future value of cashflow stream: Let the cashflow stream be $(x_0, x_1, x_2, \cdots, x_n)$. Here, the total time is divided into n periods or intervals. Here, x_0 is the cashflow at the start and x_n is the cashflow at the end. Note that x_is may be positive, negative, or zero. Also, let r be the interest rate per period (assuming that the periods are equal). For example, if the cashflow stream happens in every six months and the interest rate per annum is R, then $r = \frac{R}{2}$.



a. Compounding done per period: The future value FV is

$$\mathrm{FV} = x_0(1+r)^n + x_1(1+r)^{n-1} + x_2(1+r)^{n-2} + \cdots + x_n$$

The present value PV is

$$\mathrm{PV} = \frac{\mathrm{FV}}{(1+r)^n} = x_0 + \frac{x_1}{1+r} + \frac{x_2}{(1+r)^2} + \cdots + \frac{x_n}{(1+r)^n}$$

b. Compounding done m times per period: The future value FV is

$$\mathrm{FV} = x_0\left(1+\frac{r}{m}\right)^{mn} + x_1\left(1+\frac{r}{m}\right)^{m(n-1)} + x_2\left(1+\frac{r}{m}\right)^{m(n-2)} + \cdots + x_n$$

The present value PV is

$$\mathrm{PV} = \frac{\mathrm{FV}}{\left(1+\frac{r}{m}\right)^{mn}} = \sum_{k=0}^{n} \frac{x_k}{\left(1+\frac{r}{m}\right)^{km}}$$

c. Compounding done continuously per period: The future value FV is

$$\mathrm{FV} = x_0\mathrm{e}^{rn} + x_1\mathrm{e}^{r(n-1)} + x_2\mathrm{e}^{r(n-2)} + \cdots + x_n$$

The present value PV is

$$\mathrm{PV} = \frac{\mathrm{FV}}{\mathrm{e}^{rn}} = x_0 + \frac{x_1}{\mathrm{e}^{r}} + \frac{x_2}{e^{2r}} + \cdots + \frac{x_n}{\mathrm{e}^{rn}}$$

Let $\tilde{x} = (x_0, x_1, x_2, \cdots, x_n)$ and $\tilde{y} = (y_0, y_1, y_2, \cdots, y_m)$ be two cashflow streams. If $\mathrm{PV}(\tilde{x}) = \mathrm{PV}(\tilde{y})$, then the cashflow streams are equal.

Annuity: An annuity is a financial contract where you pay money to an insurance company or financial institution, and in return, they provide you with a steady stream of regular income

Consider an annuity that begins payment after one period from the present, paying an amount A each period for a total of n periods. The present value P, the one-period annuity payment A, the one-period interest rate r and the number of periods n are related as follows:

$$\begin{aligned} P &= \frac{A}{1+r} + \frac{A}{(1+r)^2} + \cdots + \frac{A}{(1+r)^n} \\ &= \frac{A}{1+r}\left(1 + \frac{1}{1+r} + \cdots + \frac{1}{(1+r)^{n-1}}\right) \\ &= \frac{A}{1+r}\left(\frac{1 - \frac{1}{(1+r)^n}}{1 - \frac{1}{1+r}}\right) \\ &= \frac{A}{r}\left(1 - \frac{1}{(1+r)^n}\right) \end{aligned}$$

Thus, if you invest $P = \frac{A}{r}\left(1 - \frac{1}{(1+r)^n}\right)$ amount now, then you will receive A amount for each of the n periods.

A perpetual annuity, or perpetuity, is an ongoing stream of equal cash payments that continue forever with no end date. Because the payments last forever, it has no maturity date. We can derive the relation between the present value P, the one-period annuity payment A, and the one-period interest rate r by taking $n \to \infty$ in the above formula. This gives

$$P = \frac{A}{r}$$

Thus, if you invest $P = \frac{A}{r}$ amount now, then you will receive A amount per period forever.