Discrete modelling: Recurrence relations, examples

Discrete modelling is a mathematical approach in which systems change in distinct, countable steps rather than smoothly over time. The interest payment compounded annually, quarterly, and daily are examples of discrete modelling. Other examples include insect populations whose hatching happens in a particular season and that survive only one or two seasons. There may be plants that grow from seeds in one season and die in another. There are also models with both discrete and continuous parts. For example, a particular plant grows from seeds in a particular season but dies throughout the year.

Recurrence relation/difference equations: The recurrence relation is a fundamental concept in discrete-time models. A function x on the non-negative integers is recursively defined if $x(n)$ is a function of the values of x at some or all the non-negative numbers less than n., i.e.,

$$x(n) = f(x(n-1), x(n-2), \cdots, x(0)).$$

In order for the definition to make sense, some of $x(i)$ must be specified in advance as initial values. The number of time steps that a recursion relation refers to is called its order. Thus, $x(n) = f(x(n-1))$ is a first-order and $x(n) = f(x(n-1), x(n-2), x(n-3))$ is a third-order recursion. Similarly, $x(n) = f(x(n-4))$ is a fourth-order recursion since the order is the number of time steps, not the number of prior steps. Recursive equations are easy to analyze numerically: we can calculate as many steps as needed with a pencil and paper. Recursive equations are also often called *difference equations.*

Example 1: Consider a model for crane population and suppose that the annual growth rate is r. The growth rate r depends on environmental conditions. Suppose the value of r is 0.02, -0.01 and -0.02 under best, medium and worst environmental conditions, respectively. Now we write

$$x(n) - x(n-1) = rx(n-1) \implies x(n) = ax(n-1), \quad n = 1, 2, 3, \cdots,$$

where $a = 1 + r$. Note that a may be greater, less than or equal to 1 depending on the value of r. Since this is a first-order equation, the value of $x(0)$ must be specified. Starting from $x(0) = 100$, if we compute the crane population, we observe that the population gradually increases for $r = 0.02$ and gradually decreases for $r = -0.01, -0.02$. If we run the calculations long enough, we would observe that the population becomes very large for $r = 0.02$ and extinct for negative $r = -0.01$ and -0.02.

Example 2: We have seen in the previous example that for negative r values, the solution of the difference equation

$$x(n) - x(n-1) = rx(n-1), \quad x(0) = 100,$$

gradually decreases and ultimately tends to zero for large values of n. In such cases, suppose we add 5 crane chicks each year and all of them survive. Then the difference equation becomes

$$x(n) - x(n-1) = rx(n-1) + 5 \implies x(n) = ax(n-1) + b, \quad n = 1, 2, 3, \cdots.$$

If we run the simulation starting with $x(0) = 100$, then we observe that the population stabilizes for both $r = -0.01$ and -0.02. However, the stabilized value for $r = -0.01$ is different from that of $r = -0.02$. Also, the stabilized values are independent of the initial population, i.e., we get the same stabilized values even if we start with $x(0) = 10$.

Example 3: Fibonacci considered the growth of an idealized (biologically unrealistic) rabbit population, assuming that: a newly born pair of rabbits (one male and one female) is put in a field surrounded by a wall at the beginning of a month; a newly born pair of rabbits becomes adult at the end of one month, and at the end of their second month they always produce another pair of rabbits; and rabbits never die. Fibonacci posed the rabbit math problem: how many pairs will there be in one year?

Let the interval $[n-1, n), n = 1, 2, \cdots$ denote the n-th month and $x(n)$ denote the number of rabbit pairs in the n-th month. At the end of the first month, the newly paired rabbits become adults; hence, the number of pairs is 1 in the first month, i.e., $x(1) = 1$. Since the new adult pair reproduces another pair of rabbits at the end of the second month, we have only one pair of adult rabbits in the second month, i.e., $x(2) = 1$. In the third month, the adult pair from the second month reproduces, producing a new pair, and the adult pair continues, leading to $x(3) = 2$. We can continue the same argument.

Note that any pair (new or adult) in the $(n-2)$-th month produces a new pair in n-th month. This is because a new pair of rabbits becomes adult in the $(n-1)$-th month and reproduces a new pair in the n-th month. On the other hand, the adult pair reproduces at the end of each month. Hence, $x(n-2)$ pair of $(n-2)$-th month produce $x(n-2)$ new pair in n-th month. Further, $x(n-1)$ pair of $(n-1)$-th month remains unchanged in the n-th month [since the offspring of the new pair will appear in (n+1)-th month and the offspring of the adult pair has been counted in the previous step]. Hence,

$$x(n) = x(n-1) + x(n-2), \quad x(1) = 1,\ x(2) = 1 \qquad (n \geq 3)$$

This is an example of a second-order difference equation, and the sequence of numbers $\{x_n\}$ is called the Fibonacci sequence.

For simplicity, we write $x(n)$ as x_n. Then, we write a general recurrence relation in the form $x_n = f(x_{n-1}, x_{n-2}, \cdots, x_0)$. Similarly, the Fibonacci numbers are generated from

$$x_n = x_{n-1} + x_{n-2}, \quad x_1 = 1, x_2 = 1 \qquad (n \geq 3).$$

Equilibrium/fixed point: An equilibrium point or a fixed point in a discrete dynamical system is a state that does not change over time steps when the system's update rule is applied. For a first-order system defined by a function f, where the next state is $x_{n+1} = f(x_n)$, $n \geq 0$, a point x^* is a fixed point (or equilibrium point) if it satisfies $f(x^*) = x^*$. If we start the system at $x_0 = x^*$, it stays at x^* for every future time step. Geometrically, fixed points are the locations where the graph of $y = f(x)$ intersects the straight line $y = x$.

Example: The fixed point of

$$x_{n+1} = \frac{1}{2}x_n(x_n - 1) + 1$$

is found by solving

$$x = \frac{1}{2}x(x-1) + 1 \implies x^2 - 3x + 2 = 0 \implies x = 1,\, 2$$

Hence, the recurrence relation has two fixed points $x^* = 1$ and $x^* = 2$.

Solution of linear first-order difference equation: The simplest example of first order difference equation is $x_n = ax_{n-1}$ for $n \geq 1$, which we can also write as $x_{n+1} = ax_n$ for $n \geq 0$. The equilibrium solution x^* satisfies $x^* = ax^* \implies (1-a)x^* = 0$. Hence, $x^* = 0$ is the unique equilibrium point for $a \neq 1$. For $a = 1$, every point is an equilibrium point. Now

$$\begin{aligned}
x_1 &= ax_0 \\
x_2 &= ax_1 = a^2x_0 \\
x_3 &= ax_2 = a^3x_0 \\
&\vdots \\
x_n &= a^n x_0
\end{aligned}$$

From the last relation, we see that the solution's long-term behaviour depends on a. If $|a| > 1$, then $|x_n| \to \infty$ and the equilibrium $x^* = 0$ is said to be unstable. On the other hand, if $|a| < 1$, then $x_n \to 0$ as $n \to \infty$ and then we say that the equilibrium $x^* = 0$ is asymptotically stable. Further, if $a > 0$, then x_n has the same sign as that of x_0 for all n. However, the solution alternates between positive and negative values for $a < 0$. The cases $a = 1$ and $a = -1$ are special. When $a = 1$, then $x_n = x_0$ for all n and hence every x_0 is a fixed point. On the other hand, if $a = -1$, then $x_n = (-1)^n x_0$ and thus the solution flips back and forth between x_0 and $-x_0$. If we go back to Example 1, we see that $a = 1 + r > 1$ for $r = 0.02$, and hence the crane population grows large after a sufficiently long time. However, $0 < a < 1$ for $r = -0.01$ and -0.02, which leads to extinction.

We generalized the above difference equation to $x_{n+1} = ax_n + b$, where b is a constant independent of n. The equilibrium point $x^* = \frac{b}{1-a}$ for $a \neq 1$. For $a \neq 1$, we introduce $y_n = x_n - x^*$, which gives

$$y_{n+1} = x_{n+1} - x^* = ax_n + b - x^* \implies y_{n+1} = ax_n + (1-a)x^* - x^* \implies y_{n+1} = ay_n.$$

Hence,

$$y_n = a^n y_0, \qquad y_0 = x_0 - x^*.$$

Using the arguments of the previous paragraph, we conclude that $y_n \to 0$, i.e., $x_n \to x^*$ as $n \to \infty$ for $|a| < 1$ and $|x_n - x^*| \to \infty$, i.e., $|x_n| \to \infty$ as $n \to \infty$ for $|a| > 1$. Further, if

$a > 0$, then $x_n - x^*$ has the same sign as that of $x_0 - x^*$, i.e., the sequence $\{x_n\}$ is monotonic. However, $x_n - x^*$ alternates between positive and negative values, i.e., x_n oscillates around x^* for $a < 0$. For $a = -1$, we have $x_n - x^* = (-1)^n (x_0 - x^*)$, which implies

$$x_{2k+1} - x^* = -x_0 + x^* \quad \text{and} \quad x_{2k} - x^* = x_0 - x^* \implies x_{2k+1} = x^* - (x_0 - x^*) \quad \text{and} \quad x_{2k} = x_0$$

Thus, x_n flips between x_0 and $2x^* - x_0$, which are symmetric about x^*. Finally, no fixed point exists for $a = 1$. In this

$$x_1 = x_0 + b,\ x_2 = x_1 + b = x_0 + 2b, \cdots, x_n = x_0 + nb$$

Hence, $|x_n| \to \infty$ as $n \to \infty$ when $a = 1$.

A more general form is $x_{n+1} = ax_n + b_n$, where b_n is a given sequence. This equation is nonhomogeneous due to the presence of the b_n term. The cases of $b_n = 0$ and $b_n = b =$ constant have been discussed above. When b_n varies with n, it is not possible to reduce the equation to a homogeneous form. To see its occurrence in real life, suppose P amount is deposited at the beginning of a certain year in bank with annual interest rate r compounded monthly. At the end of each month, some amount of money may be deposited/withdrawn. Let b_n be the amount deposited/withdrawn at the end of n-th month and x_n be the total amount at the end of n-th month.

$$x_n = \left(1 + \frac{r}{12}\right)^n x_{n-1} + b_n, \qquad x_0 = P$$

Here, b_n is positive for a deposit and negative for a withdrawal. Since b_n is not a constant, it cannot be converted to $y_{n+1} = ay_n$ form.