Semester II, 2017-18
Department of Physics, IIT Kanpur

PHY103A: Lecture # 10

(Text Book: Intro to Electrodynamics by Griffiths, 3rd Ed.)

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22-Jan-2018

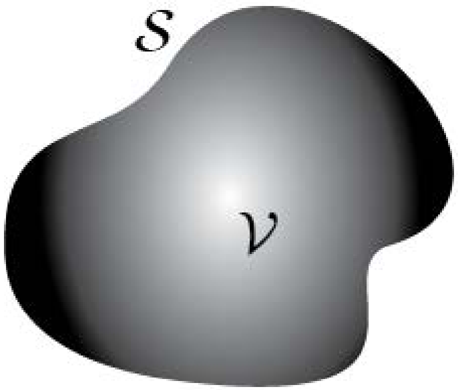
Notes

- Piazza enrollment
- TAs will be helping with Piazza queries
- Other forms of interaction remain the same

Summary of Lecture # 9:

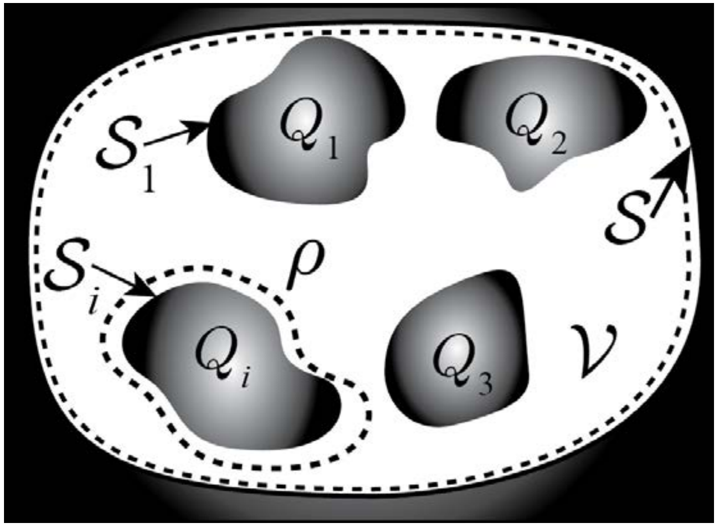
If the potential V(**r**) is a solution to the Laplace's equation then V(**r**) is the average value of potential over a spherical surface of radius *R* centered at **r**.

$$V(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_{sphere} V da$$

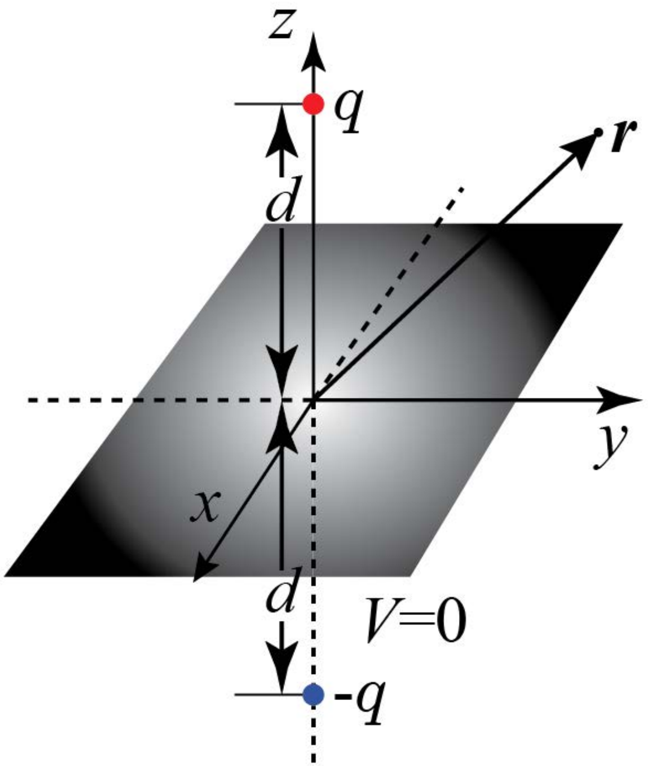


First Uniqueness Theorem: The solution to Laplace's Equation in some volume $\mathcal{V}$ is uniquely determined if V is specified on the boundary surface $\mathcal{S}$.

Second Uniqueness Theorem: In a volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the total charge on each conductor is given.



Applications of Uniqueness theorem:
(The method of images)



Q: What is the potential in the region above the infinite grounded conducting plane?

Ans: $V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}}$ × **No** **Why?**

Because the point charge will induce some charges in the conducting plane which will also contribute to the total potential

We have to solve Poisson's equation $\nabla^2 V = -\rho/\epsilon_0$ with the following boundary conditions:

(i) $V = 0$ when $z = 0$

(ii) $V \rightarrow 0$ when $x^2 + y^2 + z^2 \gg d^2$

Trick: Let's first solve the problem by removing the plate and adding $-q$ at $z = -d$

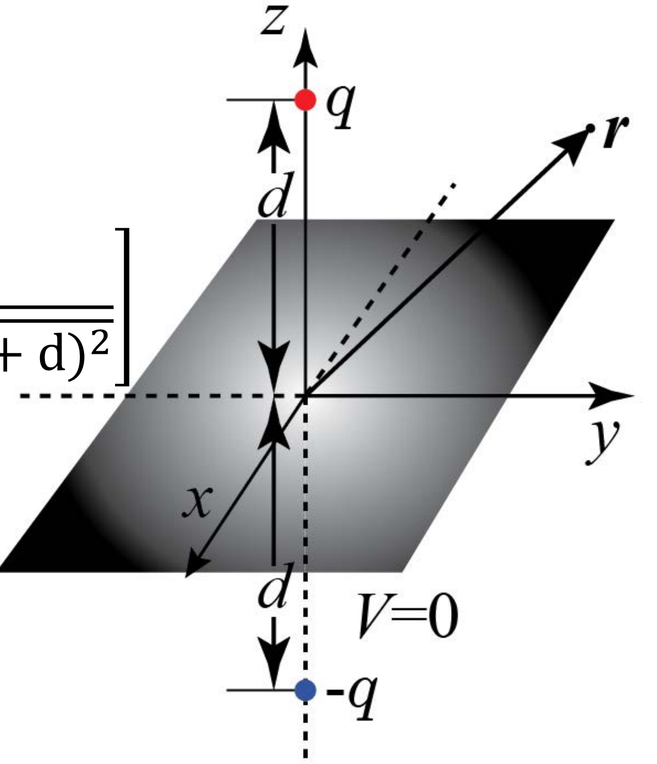
$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}}\right]$$

Does this satisfy the boundary conditions? **(i) ✓Yes** **(ii) ✓Yes**

So, the first uniqueness theorem says that this is ***the*** solution of the problem.

Applications of Uniqueness theorem:

(The method of images)



Q: What is the induced surface charge?

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

Just near conductor, we know that

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \Rightarrow -\frac{\partial V}{\partial n}\hat{\mathbf{n}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \Rightarrow \sigma = -\epsilon_0\frac{\partial V}{\partial n}$$

In the present problem, $\hat{\mathbf{n}} = \hat{\mathbf{z}}$

And, $\sigma = -\epsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0}$

$$= -\epsilon_0 \frac{q}{4\pi\epsilon_0}\left[\frac{-(z-d)}{[x^2+y^2+(z-d)^2]^{\frac{3}{2}}} + \frac{(z+d)}{[x^2+y^2+(z+d)^2]^{\frac{3}{2}}}\right]_{z=0}$$

So, $\sigma = -\frac{qd}{2\pi(x^2+y^2+d^2)^{3/2}}$ (induced charge is –ve since q is +ve)

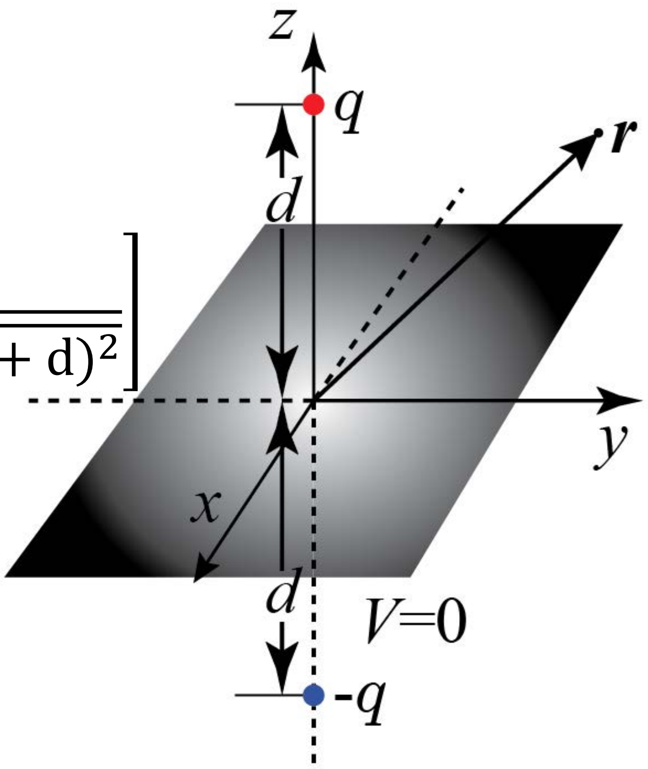
Total induced charge:

$$Q = \int \sigma da = \int_{s=0}^{\infty}\int_{\phi=0}^{2\pi}\frac{qd}{2\pi(s^2+d^2)^{3/2}} s ds d\phi = -q$$

(correct prediction)

Applications of Uniqueness theorem:

(The method of images)



Q: What is the Force and Energy?

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

The potential due to the image charge configuration is same as due to the conducting plane.

- Therefore, the electric field at any point must be the same in the two configurations.

- And therefore, the force on charge q must be the same in the two configurations.

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}\hat{\mathbf{z}}$$

Energy of the image charge system:

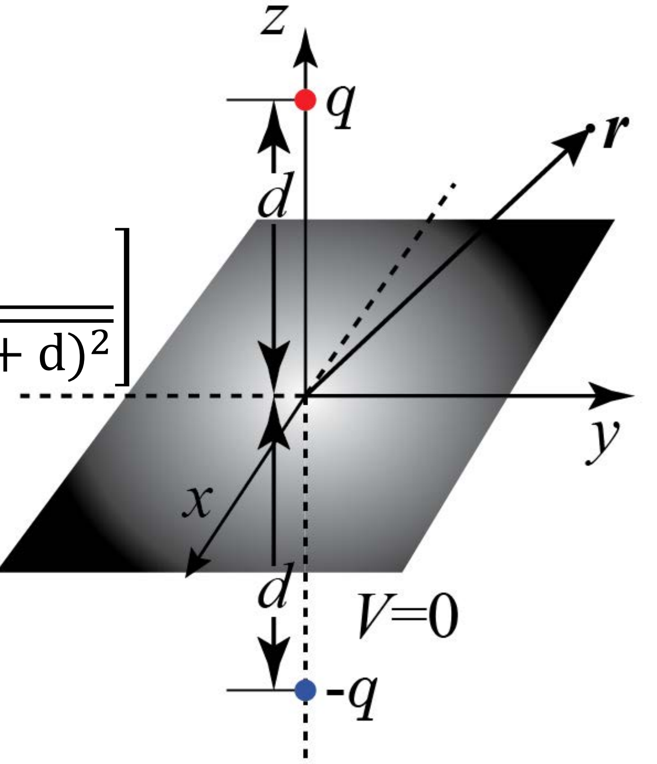
$$W = \frac{1}{2}\sum_i q_i V(\mathbf{r_i}) = \frac{1}{2}\left[q\frac{(-q)}{4\pi\epsilon_0(2d)} + (-q)\frac{q}{4\pi\epsilon_0 2d}\right] = \frac{-q^2}{8\pi\epsilon_0 d}$$

Energy of q and the conducting plane:

$$W = \frac{\epsilon_0}{2}\int E^2 d\tau = \int_\infty^d \mathbf{F}\cdot d\mathbf{l} = \int_\infty^d \frac{1}{4\pi\epsilon_0}\frac{q^2}{(2z)^2}dz = \frac{-q^2}{16\pi\epsilon_0 d}$$

(So, the electrostatic energy is not predicted correctly)

Applications of Uniqueness theorem:
(The method of images)



Q: What is the Force and Energy?

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}}\right]$$

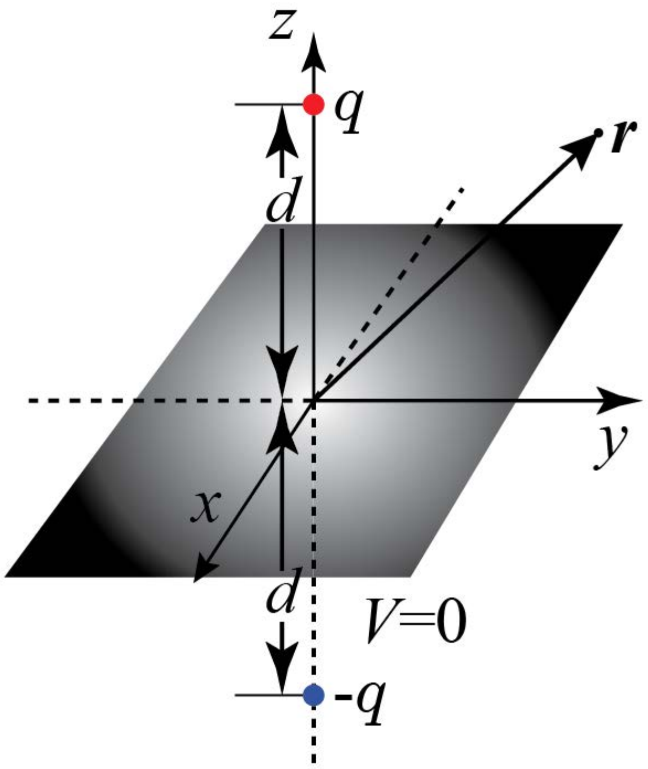
The potential due to the image charge configuration is same as due to the conducting plane.

- Therefore, the electric field at any point must be the same in the two configurations.
- And therefore, the force on charge q must be the same in the two configurations.

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}\hat{\mathbf{z}}$$

- However, the electrostatic energy of the two configurations need not be the same.

Questions 1:



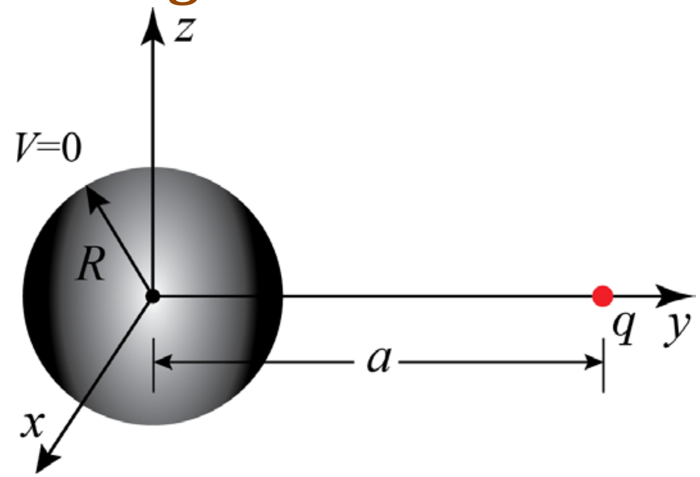
Q: What about $-2q$ at $z = -2d$ instead of $-q$ at $z = -d$

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{2}{\sqrt{x^2+y^2+(z+2d)^2}}\right]$$

Does not satisfy the boundary condition (i) $V = 0$ when $z = 0$

Applications of Uniqueness theorem: Other Image Problem:

Q: What is the potential outside the grounded conducting sphere?

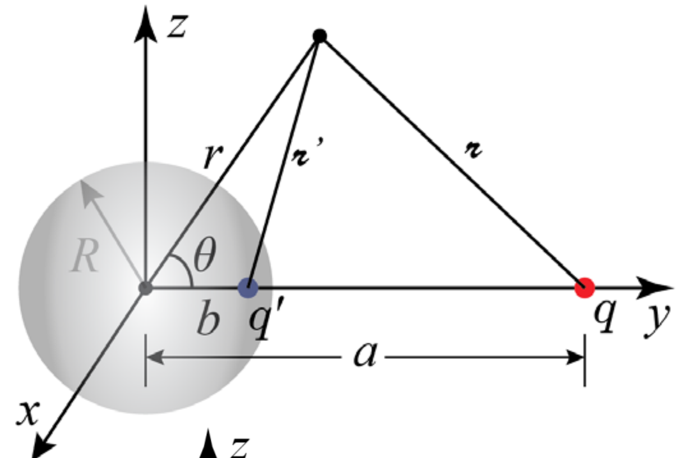


We have to solve Poisson's equation $\nabla^2 \mathrm{V} = -\rho/\epsilon_0$ with the following boundary conditions:

(i) $\mathrm{V} = 0$ at $r = R$

Trick: We first solve a different problem, with charge q and another point charge q' such that

$$q' = -\frac{R}{a} q \qquad\qquad b = \frac{R^2}{a}$$

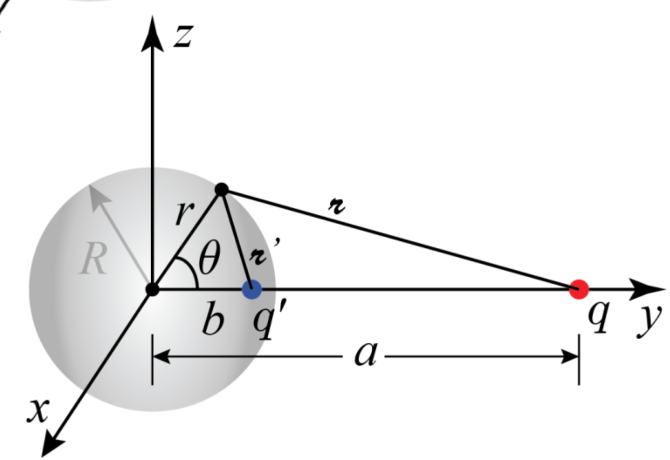


The potential at **r** is $\quad \mathrm{V}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right)$

At $\mathbf{r} = R \qquad \mathscr{r} = \sqrt{R^2 + a^2 - 2aR\cos\theta}$

$$\mathscr{r}' = \sqrt{R^2 + b^2 - 2bR\cos\theta} = \frac{R}{a}\mathscr{r}$$

$$\mathrm{V(R)} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} - \frac{Rq}{a}\frac{a}{R\,\mathscr{r}}\right) = 0$$

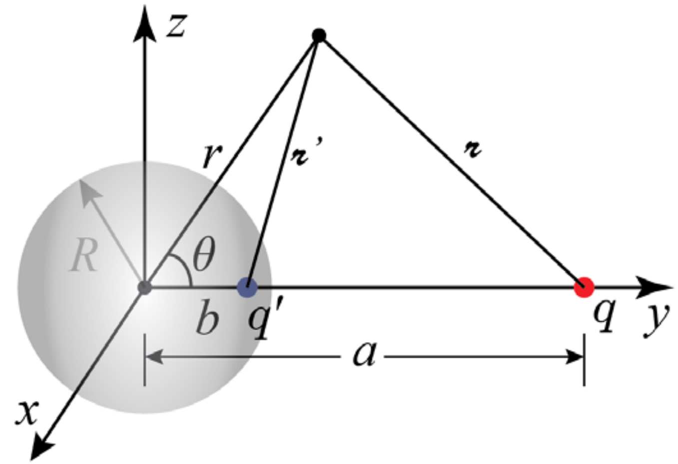


So, the potential $\mathrm{V}(\mathbf{r})$ satisfies the boundary condition.

Hence, this is ***the*** solution.

Questions 2:

Q: What is the potential inside the grounded conducting sphere?



$$q' = -\frac{R}{a}q \qquad b = \frac{R^2}{a}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right)$$

$$V(\mathbf{r} = 0) \neq 0 \; ??$$

The above solution is valid only for points outside the sphere

Solving the Laplace's Equation

- This is about solving a second-order partial differential equation.

- One of the methods is called separation of variables

- In this course, we'll not be doing it.