Semester II, 2017-18
Department of Physics, IIT Kanpur

PHY103A: Lecture # 11

(Text Book: Intro to Electrodynamics by Griffiths, 3rd Ed.)

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24-Jan-2018

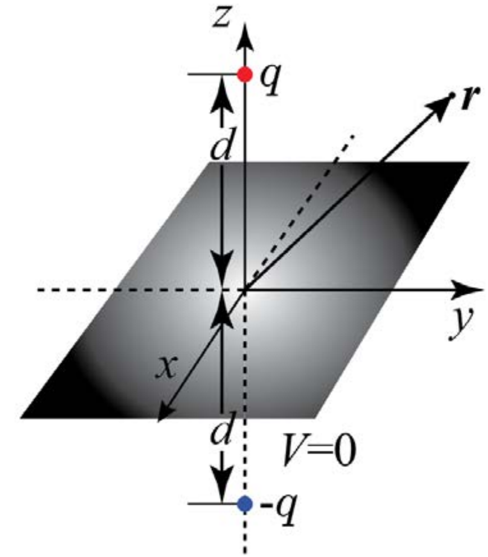
Notes

- No Class on Friday; No office hour.
- HW # 4 has been posted
- Solutions to HW # 5 have been posted

Summary of Lecture # 10:

The potential in the region **above** the infinite grounded conducting plane?

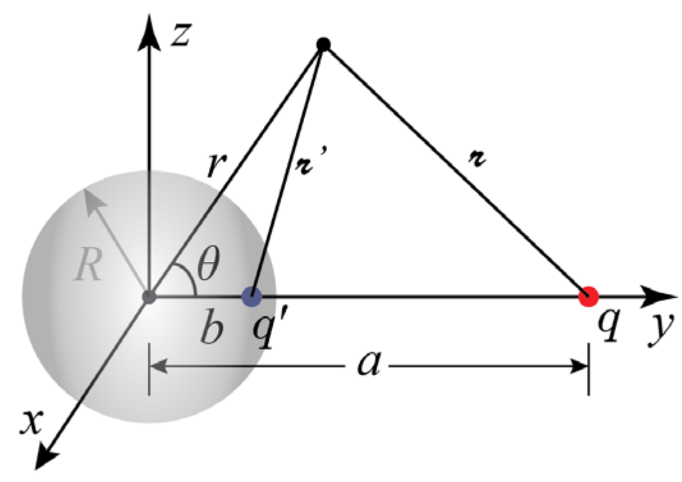
$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}}\right]$$



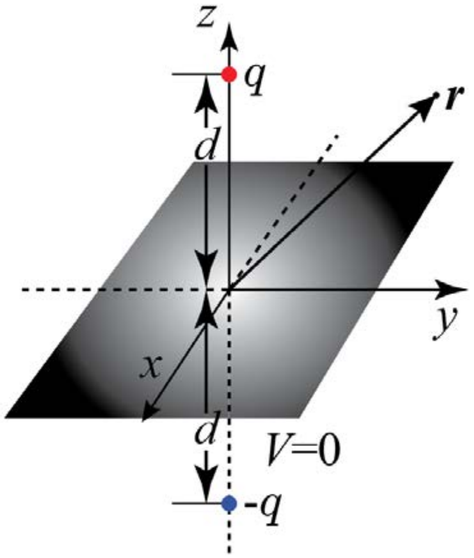
The potential **outside** the grounded conducting sphere?

$$q' = -\frac{R}{a}q \qquad b = \frac{R^2}{a}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right)$$



Question:



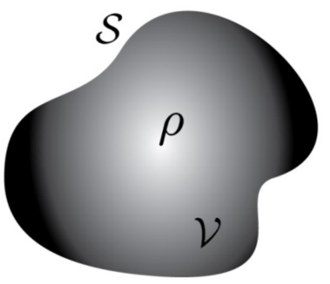
The potential in the region **above** the infinite grounded conducting plane?

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

Q: Is $kV(\mathbf{r})$ also a solution, where k is a constant?

Ans: No

Corollary to First Uniqueness Theorem: The potential in a volume is uniquely determined if (a) the charge desity throughout the region and (b) the value of V at all boundaries, are specified.



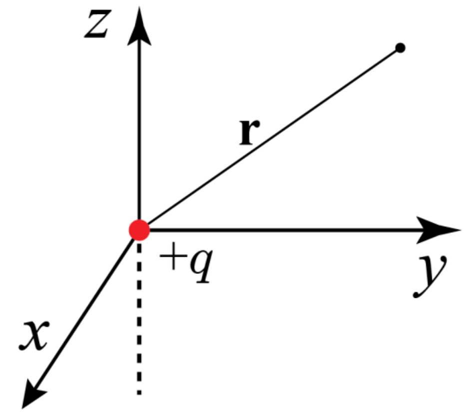
In this case one has to satisfy the Poisson's equation. But $kV(\mathbf{r})$ does not satisfy it.

$$\nabla^2 V_1 = -\frac{\rho}{\epsilon_0}$$

Exception: Only when $\rho = 0$ (everywhere), $kV(\mathbf{r})$ can be a solution as well

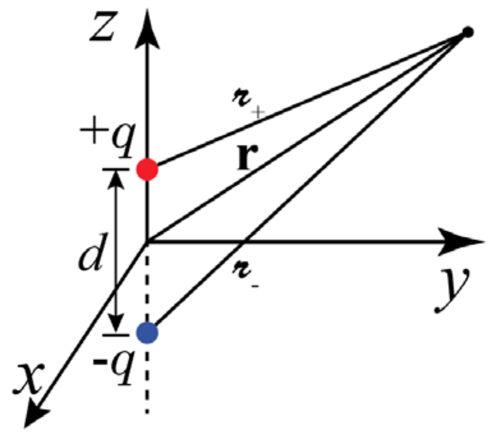
But then since in this case $V(\mathbf{r}) = \mathbf{0}$, it is a trivial solution.

Multipole Expansion (Potentials at large distances)



- What is the potential due to a point charge (monopole)?

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$ **(goes like $1/r$)**



- What is the potential due to a dipole at large distance?

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right)$$

$$\mathscr{r}_\pm^2 = r^2 + \left(\frac{d}{2}\right)^2 \mp rd\cos\theta \ = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

$$\approx r^2\left(1 \mp \frac{d}{r}\cos\theta\right) \quad (\text{for } r \gg d)$$

for $r \gg d$, and using binomial expansion

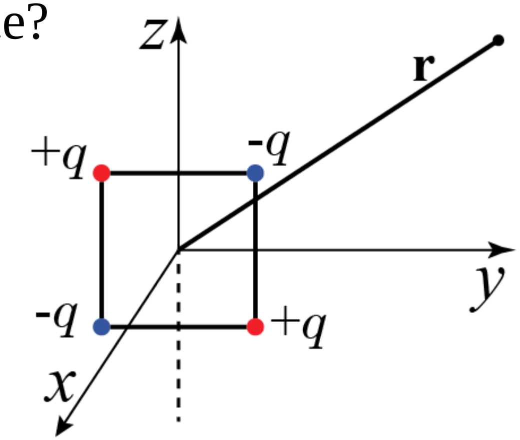
$$\frac{1}{\mathscr{r}_\pm} = \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-\frac{1}{2}} \approx \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

So, $$\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} = \frac{d}{r^2}\cos\theta$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}$$ **(goes like $1/r^2$ at large r)**

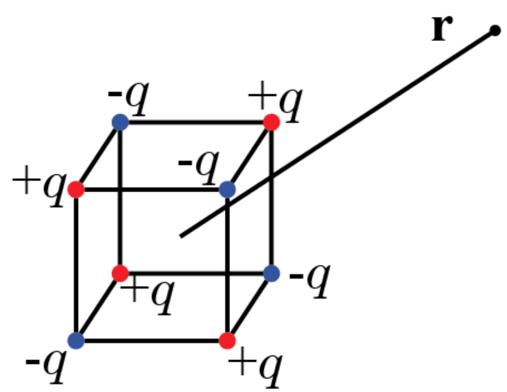
Multipole Expansion (Potentials at large distances)

- What is the potential due to a quadrupole at large distance?



(goes like $1/r^3$ at large r)

- What is the potential due to a octopole at large distance?



(goes like $1/r^4$ at large r)

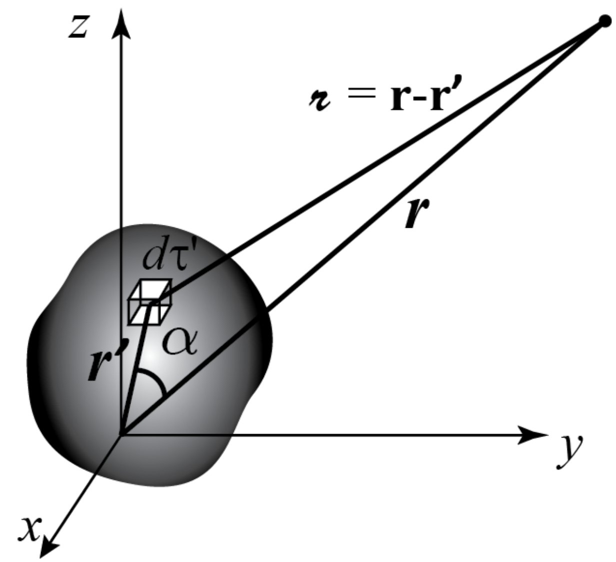
Note1: Multipole terms are defined in terms of their r dependence, not in terms of the number of charges.

Note 2: The dipole potential need not be produced by a two-charge system only . A general n-charge system can have any multipole contribution.

Multipole Expansion (Potentials at large distances)

- What is the potential due to a localized charge distribution?

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}} \quad = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau'$$



Using the cosine rule,

$$\mathscr{r}^2 = r^2 + r'^2 - 2rr'\cos\alpha$$

$$\mathscr{r}^2 = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]$$

$$\mathscr{r} = r\sqrt{1 + \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)}$$

$$\mathscr{r} = r\sqrt{1+\epsilon}$$

Source coordinates: (r', θ', ϕ')
Observation point coordinates: (r, θ, ϕ)
Angle between **r** and **r**′: α

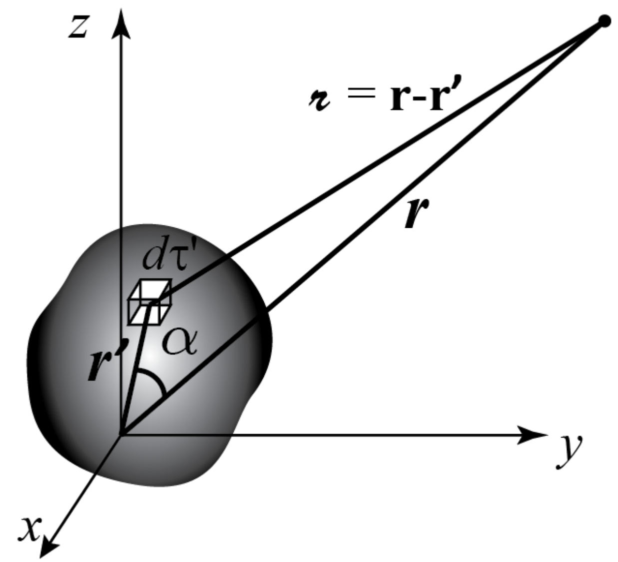
Define: $\epsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)$

So, $\dfrac{1}{\mathscr{r}} = \dfrac{1}{r}(1+\epsilon)^{-1/2}$

Or, $\dfrac{1}{\mathscr{r}} = \dfrac{1}{r}\left(1 - \dfrac{1}{2}\epsilon + \dfrac{3}{8}\epsilon^2 - \dfrac{5}{16}\epsilon^3 + \cdots\right)$ (using binomial expansion)

Multipole Expansion (Potentials at large distances)

- What is the potential due to a localized charge distribution?



$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}} \quad = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}d\tau'$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \cdots\right)$$

$$= \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2 - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\alpha\right)^3 + \cdots\right]$$

$$= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)(\cos\alpha) + \left(\frac{r'}{r}\right)^2 (3\cos^2\alpha - 1)/2 \; - \left(\frac{r'}{r}\right)^3 (5\cos^3\alpha - 3\cos\alpha)/2 \; + \cdots\right]$$

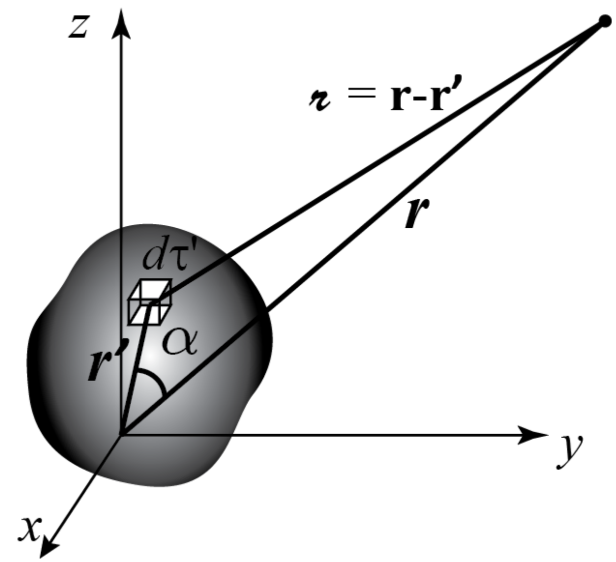
$$= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

$P_n(\cos\alpha)$ are Legendre polynomials

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}d\tau' = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')d\tau'$$

Multipole Expansion (Potentials at large distances)

- What is the potential due to a localized charge distribution?



$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}} \quad = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}d\tau'$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')d\tau'$$

$$= \frac{1}{4\pi\epsilon_0 r}\int \rho(\mathbf{r}')d\tau' + \frac{1}{4\pi\epsilon_0 r^2}\int r'(\cos\alpha)\rho(\mathbf{r}')d\tau' + \frac{1}{4\pi\epsilon_0 r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')d\tau' + \cdots$$

Monopole potential ($1/r$ dependence)

Dipole potential ($1/r^2$ dependence)

Quadrupole potential ($1/r^3$ dependence)

Multipole Expansion of V(r)

Multipole Expansion (Few comments)

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r}\int \rho(\mathbf{r}')d\tau' + \frac{1}{4\pi\epsilon_0 r^2}\int r'(\cos\alpha)\rho(\mathbf{r}')d\tau' + \frac{1}{4\pi\epsilon_0 r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')d\tau' + \cdots$$

Monopole potential ($1/r$ dependence)

Dipole potential ($1/r^2$ dependence)

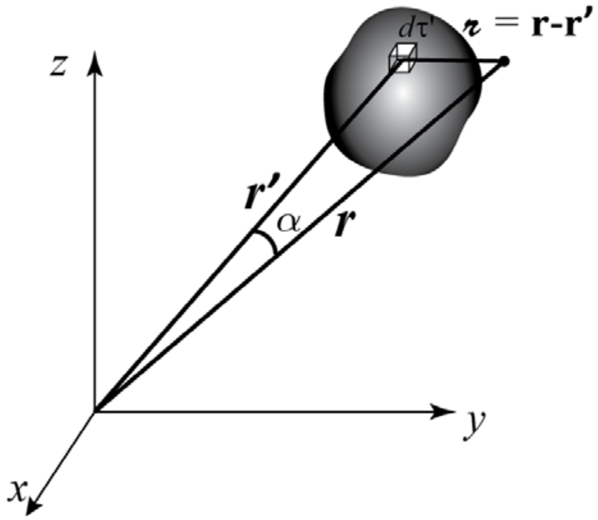
Quadrupole potential ($1/r^3$ dependence)

- **It is an exact expression, not an approximation.**
- **A particular term in the expansion is defined by its r dependence**
- **At large r, the potential can be approximated by the first non-zero term.**
- **More terms can be added if greater accuracy is required**

Questions 1:

Q: In this following configuration, is the "large **r**" limit valid, since the source dimensions are much smaller than **r**?

Ans: No. The "large **r**" limit essentially mean $|\mathbf{r}| \gg |\mathbf{r}'|$. In majority of the situations, the charge distribution is centered at the origin and therefore the "large **r**" limit is the same as source dimension being smaller than **r**.



Multipole Expansion (Monopole and Dipole terms)

Monopole term:

$$V_{\text{mono}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(\mathbf{r}')d\tau' = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

- $Q = \int \rho(\mathbf{r}')d\tau'$ is the total charge
- If $Q = 0$, monopole term is zero.
- For a collection of point charges

$$Q = \sum_{i=1}^{n} q_i$$

Dipole term:

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'(\cos\alpha)\rho(\mathbf{r}')d\tau'$$

α is the angle between $\mathbf{r}$ and $\mathbf{r}'$.

So, $r'(\cos\alpha) = \hat{\mathbf{r}} \cdot \mathbf{r}'$

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}} \cdot \int \mathbf{r}'\rho(\mathbf{r}')d\tau'$$

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

- $\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')d\tau'$ is called the dipole moment of a charge distribution
- If $\mathbf{p} = 0$, dipole term is zero.
- For a collection of point charges.

$$\mathbf{p} = \sum_{i=1}^{n} \mathbf{r_i}' q_i$$

Multipole Expansion (Monopole and Dipole terms)

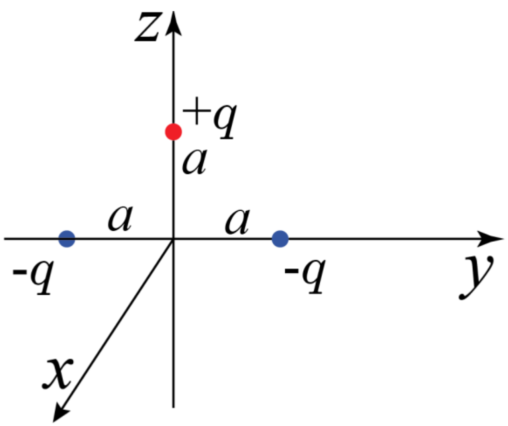
Monopole term:

$$V_{\text{mono}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(\mathbf{r}')d\tau' \quad \rightarrow \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$ (for point charges) $$Q = \sum_{i=1}^{n} q_i$$

Dipole term:

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'(\cos\alpha)\rho(\mathbf{r}')d\tau' \quad \rightarrow \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$ (for point charges) $$\mathbf{p} = \sum_{i=1}^{n} \mathbf{r_i}' q_i$$

Example: A three-charge system



$$Q = \sum_{i=1}^{n} q_i \quad = -q$$

$$\mathbf{p} = \sum_{i=1}^{n} \mathbf{r_i}' q_i = qa\,\hat{\mathbf{z}} + [-qa - q(-a)]\hat{\mathbf{y}} = qa\,\hat{\mathbf{z}}$$

Therefore the system will have both monopole and dipole contributions

Multipole Expansion (Monopole and Dipole terms)

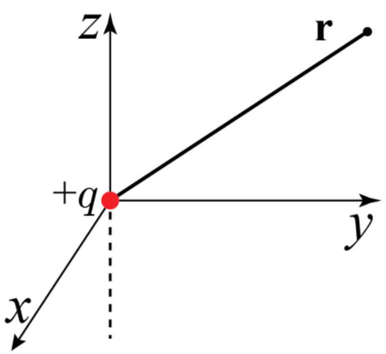
Monopole term:

$$V_{\text{mono}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(\mathbf{r}')d\tau' \quad \rightarrow \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$ (for point charges) $$Q = \sum_{i=1}^{n} q_i$$

Dipole term:

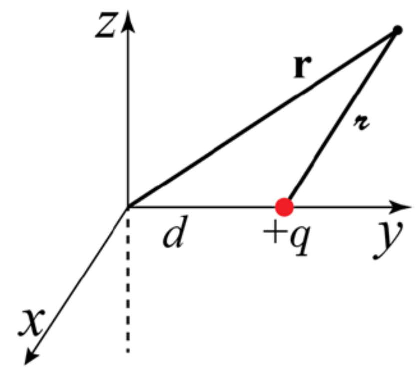
$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'(\cos\alpha)\rho(\mathbf{r}')d\tau' \quad \rightarrow \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$ (for point charges) $$\mathbf{p} = \sum_{i=1}^{n} \mathbf{r_i}' q_i$$

Example: Origin of Coordinates



$$Q = q \qquad V_{\text{mono}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

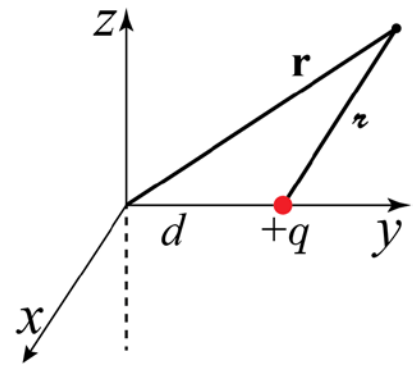
$$\mathbf{p} = 0 \qquad V_{\text{dip}}(\mathbf{r}) = 0$$



$$Q = q \qquad V_{\text{mono}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \neq \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}$$

$$\mathbf{p} = qd\hat{\mathbf{y}} \qquad V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

Questions 2:



$$Q = q \qquad V_{\text{mono}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \neq \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}$$

$$\mathbf{p} = qd\hat{\mathbf{y}} \qquad V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

Q: Why not calculate the potential directly ??

Ans: Yes, that is what should be done. For a point charge, we don't need a multipole expansion to find the potential. This is only for illustrating the connection.

The electric field of pure dipole ($Q = 0$)

$Q = 0$ And $\mathbf{p} \neq 0$ Assume $\mathbf{p} = p\hat{\mathbf{z}}$

$$\mathrm{V_{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{p\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}$$

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\mathrm{V}$$

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0$$

$$\mathbf{E}_{\mathrm{dip}}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta})$$