Semester II, 2017-18
Department of Physics, IIT Kanpur

PHY103A: Lecture # 12

(Text Book: Intro to Electrodynamics by Griffiths, 3rd Ed.)

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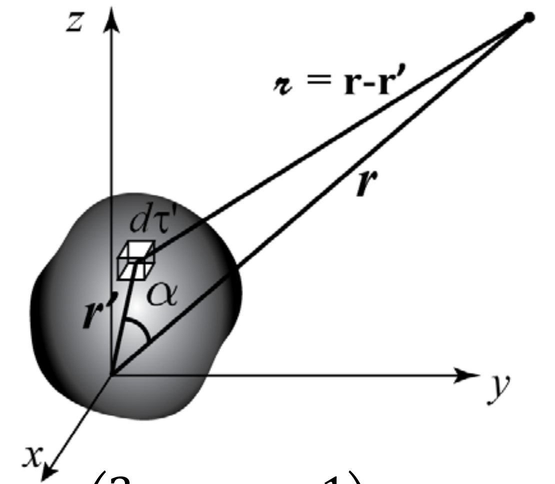
Notes

- Tutorial on Tuesday (Section C: Dr. Suratna Das)
- Lecture on Wednesday (Prof. H. Wanare)

Summary of Lecture # 11:

What is the potential due to a localized charge distribution at large **r** ?



$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}} \quad = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau'$$

$$= \frac{1}{4\pi\epsilon_0 r}\int \rho(\mathbf{r}')d\tau' + \frac{1}{4\pi\epsilon_0 r^2}\int r'(\cos\alpha)\rho(\mathbf{r}')d\tau' + \frac{1}{4\pi\epsilon_0 r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')d\tau' + \cdots$$

Monopole potential ($1/r$ dependence)

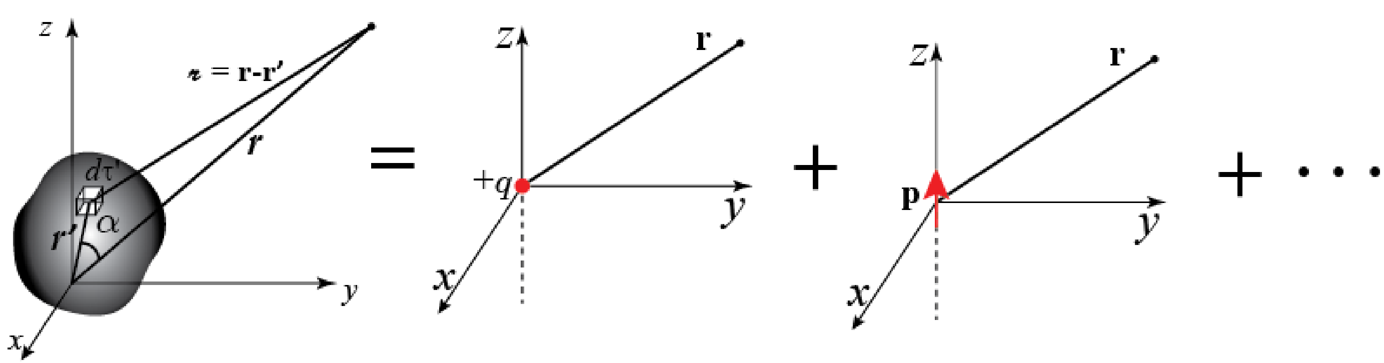
$$\boxed{Q = \int \rho(\mathbf{r}')d\tau'}$$
Total Charge

Dipole potential ($1/r^2$ dependence)

$$\boxed{\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')d\tau'}$$
Dipole moment

Quadrupole potential ($1/r^3$ dependence)

? ? ?
Quadrupole moment



The electric field of pure dipole ($Q = 0$)

$Q = 0$ And $\mathbf{p} \neq 0$ Assume $\mathbf{p} = p\hat{\mathbf{z}}$

$$\mathrm{V}_{\mathrm{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{p\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}$$

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\mathrm{V}$$

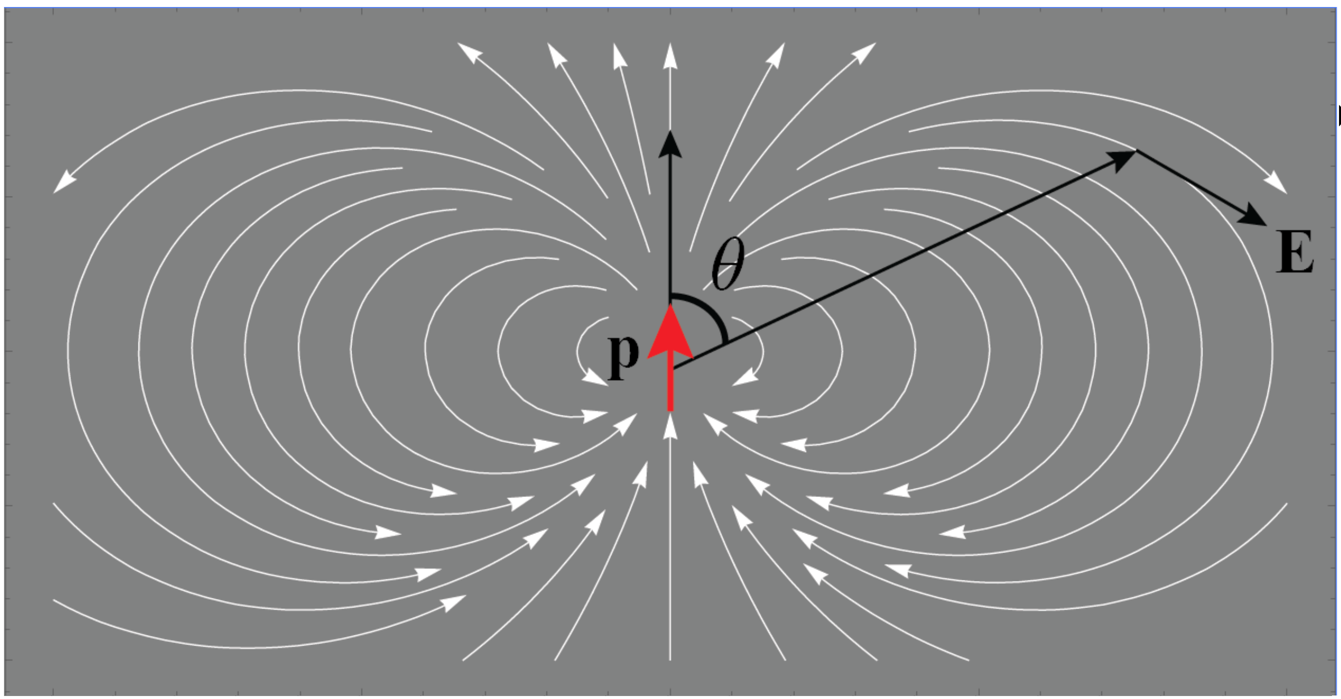
$$\left.\begin{aligned}
E_r &= -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3} \\
E_\theta &= -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3} \\
E_\phi &= -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0
\end{aligned}\right\} \quad \mathbf{E}_{\mathrm{dip}}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta})$$

The electric field of pure dipole ($Q = 0$)

$Q = 0$ And $\mathbf{p} \neq 0$ Assume $\mathbf{p} = p\hat{\mathbf{z}}$

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{p\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}$$

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}V$$



$s\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta})$

Quick Summary:

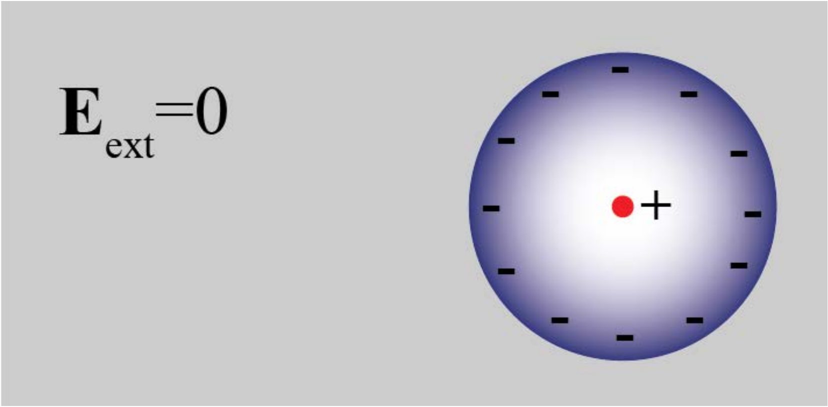
- We studied electrostatics in vacuum as well as in conductors
- Calculating Electric field $\mathbf{E}$ given a charge ρ is one of main aims of electrostatics
- Electric field can be calculated using Coulomb's law but usually it is very difficult
- The easier way is to first calculate the electric potential and then $\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\mathrm{V}$
- Electric potential can be calculated in two different ways
  - ❑ Using the differential (Poisson's) equation $\boldsymbol{\nabla}^2\mathrm{V} = -\dfrac{\rho}{\epsilon_0}$
    - ✓ Uniqueness theorems guarantee that a solution is unique if it is found
    - ✓ We studied method of images to be able to guess a solution in some cases
    - ✓ Poisson's equation can sometimes be analytically solved
  - ❑ Using the integral equation $\mathrm{V}(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \frac{dq}{\mathscr{r}}$
    - ✓ Multipole expansion method is based on using this form.
    - ✓ Using multipole expansion, approximate potential at large $\mathbf{r}$ are calculated

Electrostatics in Matter (Electric Fields in Matter)

Polarization (Induced Dipoles)

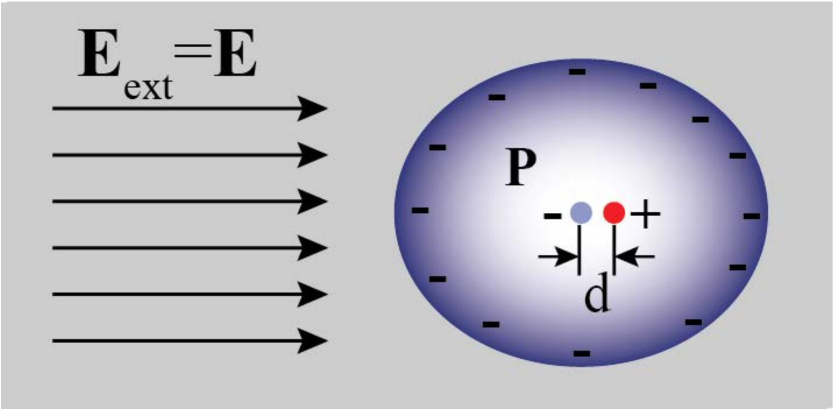
A neutral atom

Total charge $Q = 0$; Dipole moment $\mathbf{p} = 0$;



Atom in an electric field

Total charge $Q = 0$; Dipole moment $\mathbf{p} \neq 0$;

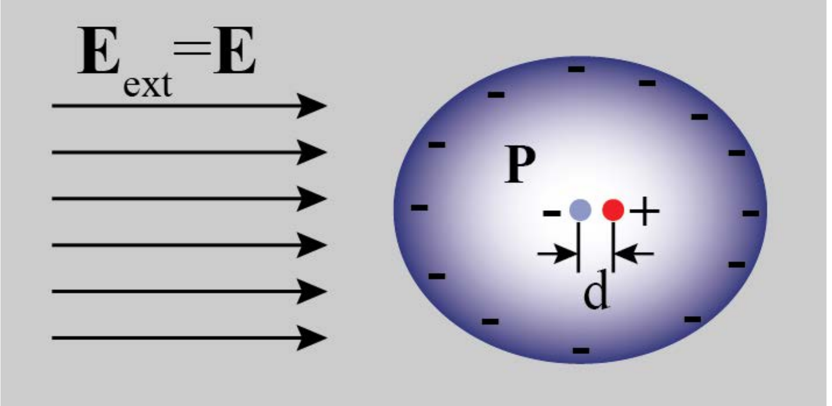


Atom has become polarized

$$\boxed{\mathbf{p} = \alpha\, \mathbf{E}}$$

- α is called the atomic polarizability
- α depends on the detailed structure of the atom
- α is determined experimentally

Questions 1:



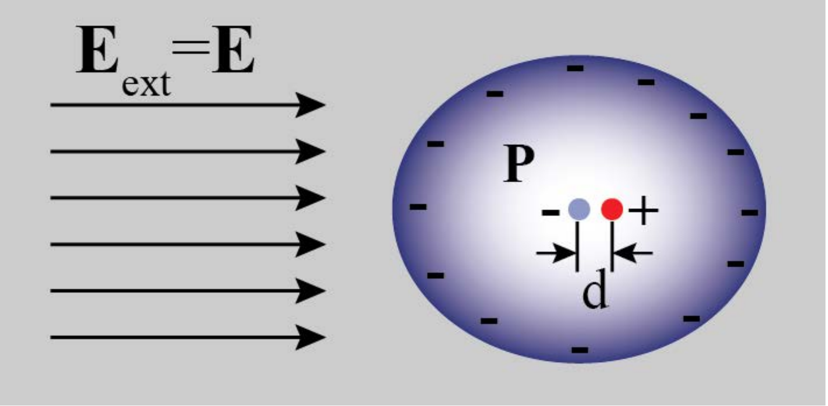
$$\mathbf{p} = \boldsymbol{\alpha}\,\mathbf{E}\ ?$$

Q: How do we know that induced dipole moment is proportional to the electric field, only in a linear manner and not in quadratic or any other manner:

Ans: We know that dipole moment is linearly proportional to the separation between charges. With weak electric fields, the separation is found to change linearly with the field amplitude and that is what gives the above relationship. However, at strong enough field amplitudes, it is found that the dipole moment not only has a linear dependence but also the quadratic and so on. This regime of physics is called the non-linear physics. This is analogous to what happens in a simple harmonic oscillator.

Polarization (Induced Dipoles)

Ex. 4.1 (Griffiths, 3rd Ed.): An atom can be considered as a point nucleus of charge $+q$ surrounded by a uniformly charged spherical electron cloud of charge $-q$ and radius a ? Find the atomic polarizability.



Here, we are assuming that the electron cloud remains spherical in shape even in the presence of the external electric field.

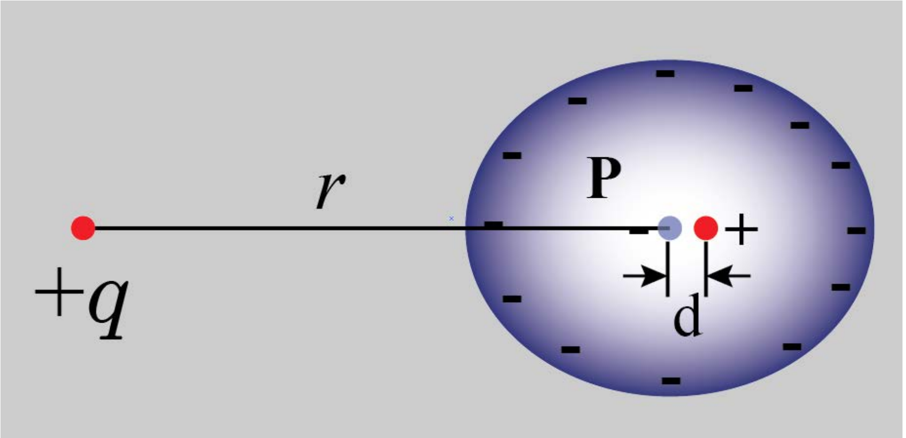
In equilibrium, the field at the nucleus due to the electron cloud is

$$E_{\text{elec}} = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3} = E \quad \Rightarrow \quad d = \frac{4\pi\epsilon_0\, a^3 E}{q}$$

So the atomic polarizability $\alpha = \frac{p}{E} = \frac{qd}{E} = \frac{4\pi\epsilon_0\, a^3 E}{E} = 4\pi\epsilon_0\, a^3$

Polarization (Induced Dipoles)

Ex. 4.4 (Griffiths, 3rd Ed.): A point charge q is situated a large distance r from a neutral atom of polarizability α. Find the force of attraction between them.



The field due to charge q is

$$\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

The induced dipole moment is

$$\mathbf{p} = \alpha\,\mathbf{E} = \frac{\alpha}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

The field due to this dipole at the location ($\theta = \pi$) of the charge is

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta}) = \frac{1}{4\pi\epsilon_0 r^3}\frac{\alpha}{4\pi\epsilon_0}\frac{q}{r^2}(-2\,\hat{\mathbf{r}})$$

Therefore, the force is $\quad \mathbf{F} = q\mathbf{E}_{\text{dip}}(\mathbf{r}) = -2\alpha\left(\frac{q}{4\pi\epsilon_0}\right)^2\frac{1}{r^5}\,\hat{\mathbf{r}}$

Polarization (Induced Dipoles)

Polarizability Tensor

In the simplest of the situations the induced dipole moment **p** is proportional to the applied electric field **E** and the constant of proportionality is α.

$$\mathbf{p} = \alpha\, \mathbf{E}$$

However, in general α is a tensor. This means that, in general, all the three components of the field can induce dipole moment in any of the three directions.

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{pmatrix}}_{\text{Polarizability Tensor}} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

But, one can always diagonalize the tensor and find a coordinate axes (x_d, y_d, z_d) such that induced dipole moment in a given direction is proportional to the electric field in the same direction.

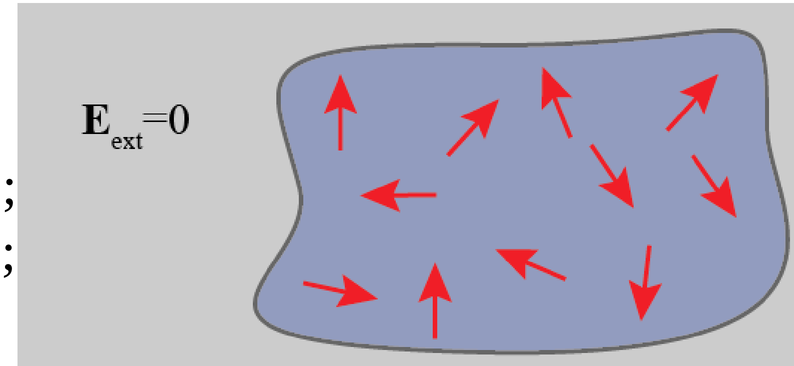
$$\begin{pmatrix} p_{x_d} \\ p_{y_d} \\ p_{z_d} \end{pmatrix} = \begin{pmatrix} \alpha_{x_d} & 0 & 0 \\ 0 & \alpha_{y_d} & 0 \\ 0 & 0 & \alpha_{z_d} \end{pmatrix} \begin{pmatrix} E_{x_d} \\ E_{y_d} \\ E_{z_d} \end{pmatrix}$$

Polarization (Permanent Dipoles)

A neutral atom has no dipole moment to begin with but some molecules (polar molecules) have permanent dipole moment, even without the external electric field.

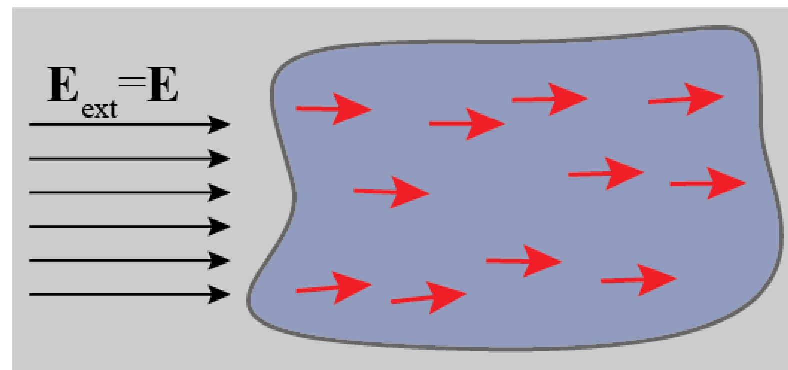
<u>Polar molecules</u>

Total charge $Q = 0$;
Net dipole moment $\mathbf{p} = 0$;



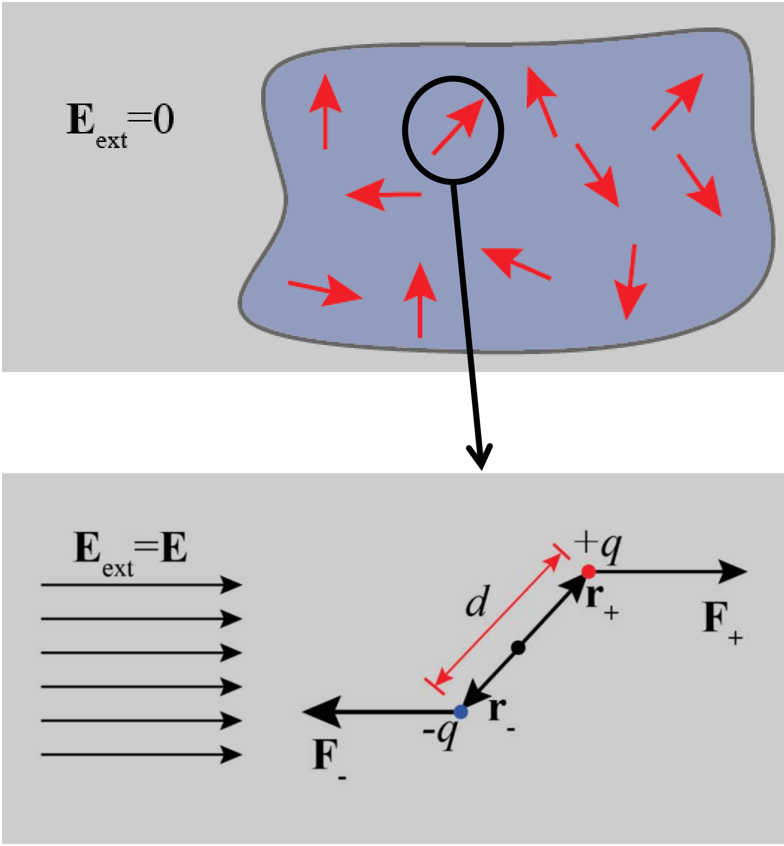
<u>Polar molecules in an electric field</u>

Total charge $Q = 0$;
Net dipole moment $\mathbf{p} \neq 0$;



Molecules already had dipole moments
The dipole moments have now become aligned

Polarization (Permanent Dipoles)



Uniform electric field E: $\mathbf{F}_+ = -\mathbf{F}_-$.

The net force on a molecule is zero $\boxed{\mathbf{F} = 0}$

The net torque on a molecule is

$$\mathbf{N} = (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-)$$

$$= [(\mathbf{d}/2) \times (q\mathbf{E})] + [(-\mathbf{d}/2) \times (-q\mathbf{E})]$$

Or, $\boxed{\mathbf{N} = \mathbf{p} \times \mathbf{E}}$

In a non-Uniform electric field E

$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- \quad = q(\mathbf{E}_+ + \mathbf{E}_-) \quad = q(\Delta\mathbf{E})$$

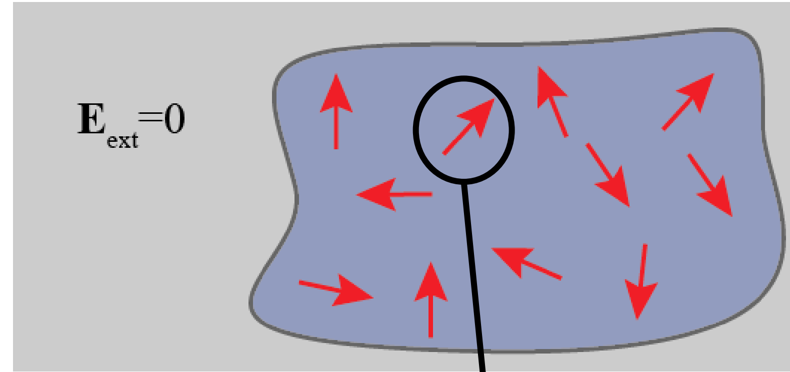
$$\Delta\mathbf{E} = \Delta E_x\hat{\mathbf{x}} + \Delta E_y\hat{\mathbf{y}} + \Delta E_z\hat{\mathbf{z}}$$

$$\Delta E_x \to \Delta E_x(x,y,z) = \nabla E_x(x,y,z) \cdot \mathbf{d}$$ (Using $\mathrm{d}T(x,y,z) = \nabla T(x,y,z) \cdot d\mathbf{l}$)

$$\Delta E_x = \nabla E_x \cdot \mathbf{d} = x\frac{\partial}{\partial x}E_x + y\frac{\partial}{\partial y}E_x + z\frac{\partial}{\partial z}E_x$$

$$\Delta\mathbf{E} = \Delta E_x\hat{\mathbf{x}} + \Delta E_y\hat{\mathbf{y}} + \Delta E_z\hat{\mathbf{z}} = (\nabla E_x \cdot \mathbf{d})\hat{\mathbf{x}} + \left(\nabla E_y \cdot \mathbf{d}\right)\hat{\mathbf{y}} + (\nabla E_z \cdot \mathbf{d})\hat{\mathbf{z}}$$

Polarization (Permanent Dipoles)



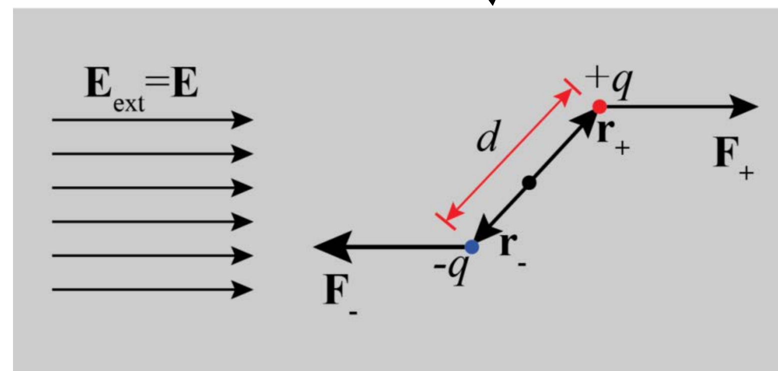
<u>Uniform electric field E:</u> $\mathbf{F}_+ = -\mathbf{F}_-$.

The net force on a molecule is zero $\boxed{\mathbf{F} = 0}$

The net torque on a molecule is

$$\begin{aligned}\mathbf{N} &= (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-)\\ &= [(\mathbf{d}/2) \times (q\mathbf{E})] + [(-\mathbf{d}/2) \times (-q\mathbf{E})]\end{aligned}$$

Or, $\boxed{\mathbf{N} = \mathbf{p} \times \mathbf{E}}$



<u>In a non-Uniform electric field E</u>

$$\Delta\mathbf{E} = (\nabla E_x \cdot \mathbf{d})\hat{\mathbf{x}} + (\nabla E_y \cdot \mathbf{d})\hat{\mathbf{y}} + (\nabla E_z \cdot \mathbf{d})\hat{\mathbf{z}}$$

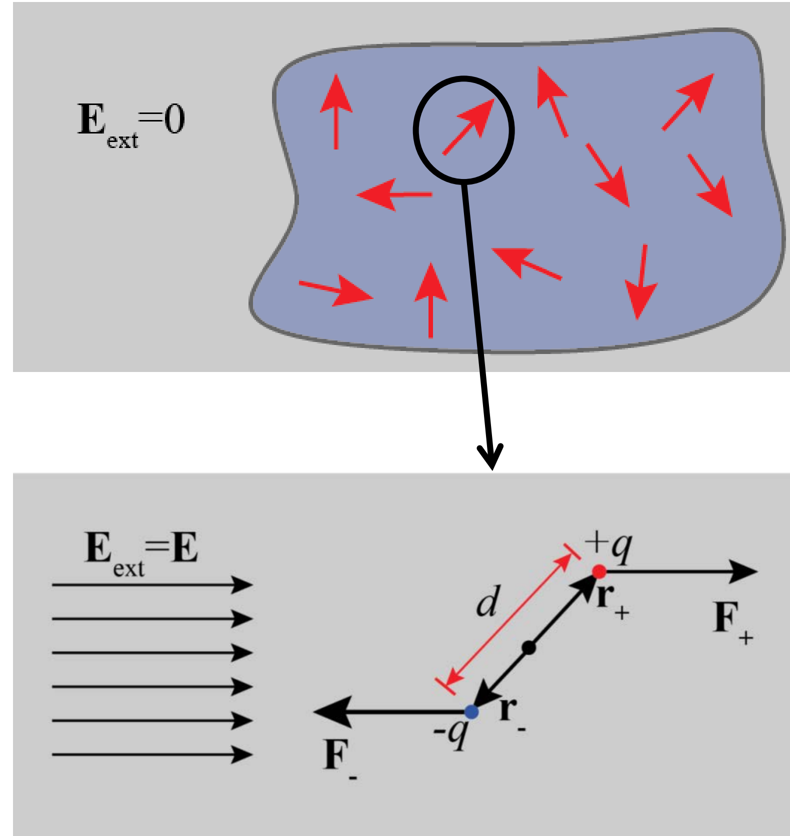
$$= \left(x\frac{\partial}{\partial x}E_x + y\frac{\partial}{\partial y}E_x + z\frac{\partial}{\partial z}E_x\right)\hat{\mathbf{x}} + \left(x\frac{\partial}{\partial x}E_y + y\frac{\partial}{\partial y}E_y + z\frac{\partial}{\partial z}E_y\right)\hat{\mathbf{y}}$$

$$+ \left(x\frac{\partial}{\partial x}E_z + y\frac{\partial}{\partial y}E_z + z\frac{\partial}{\partial z}E_z\right)\hat{\mathbf{z}}$$

$$= x\frac{\partial}{\partial x}(E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}) + y\frac{\partial}{\partial y}(E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}) + z\frac{\partial}{\partial z}(E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}})$$

$$= x\frac{\partial}{\partial x}\mathbf{E} + y\frac{\partial}{\partial y}\mathbf{E} + z\frac{\partial}{\partial z}\mathbf{E} = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\mathbf{E} = (\mathbf{d}\cdot\nabla)\mathbf{E}$$

Polarization (Permanent Dipoles)



<u>Uniform electric field E:</u> $\mathbf{F}_+ = -\mathbf{F}_-$.

The net force on a molecule is zero $\boxed{\mathbf{F} = 0}$

The net torque on a molecule is

$$\mathbf{N} = (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-)$$
$$= [(\mathbf{d}/2) \times (q\mathbf{E})] + [(-\mathbf{d}/2) \times (-q\mathbf{E})]$$

Or, $\boxed{\mathbf{N} = \mathbf{p} \times \mathbf{E}}$

<u>In a non-Uniform electric field E</u>

$$\Delta\mathbf{E} = (\mathbf{d} \cdot \nabla)\mathbf{E}$$

$\mathbf{F} = q\Delta\mathbf{E}$ Or, $\boxed{\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}}$

The net torque on a molecule is

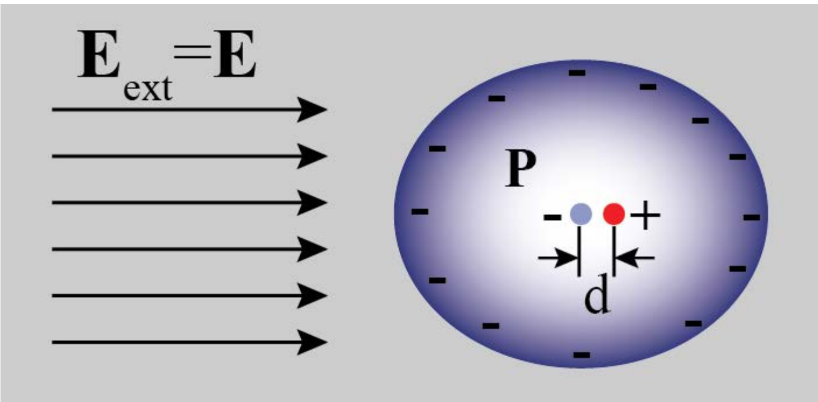
$\boxed{\mathbf{N} = (\mathbf{p} \times \mathbf{E})}$ (about the center of the dipole)

$\boxed{\mathbf{N} = (\mathbf{p} \times \mathbf{E}) + (\mathbf{r} \times \mathbf{F})}$ (about a point **r** away from the center of the dipole)

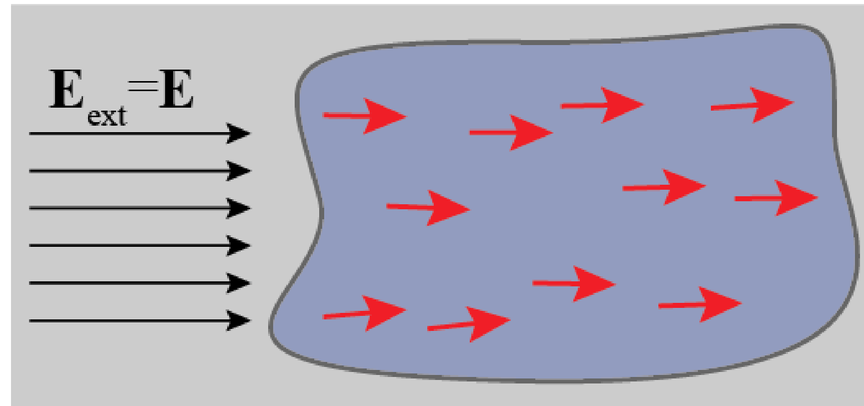
Polarization in a medium

Two mechanism for polarization in a medium

Induced Dipoles



Alignment of permanent dipoles



The two mechanisms need not be independent.

A permanent dipole may also get some dipole moment induced.

Total dipole moment of per unit volume is called the polarization **P**.

$$\mathbf{P} \equiv \text{dipole moment per unit volume}$$