Semester II, 2017-18
Department of Physics, IIT Kanpur

PHY103A: Lecture # 13

(Text Book: Intro to Electrodynamics by Griffiths, 3rd Ed.)

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Summary of Lecture # 12:

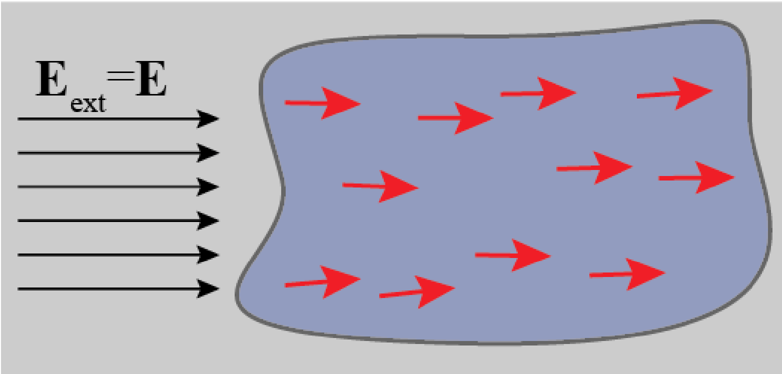
- Electric Fields in Matter

1. Induced Polarization $\mathbf{p} = \alpha\,\mathbf{E}$ $\boldsymbol{\alpha}$ is called the atomic polarizability

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

Polarizability Tensor

2. Permanent Polarization (Polar molecules)



Uniform electric field E

$\mathbf{F} = 0$ $\mathbf{N} = \mathbf{p} \times \mathbf{E}$

In a non-Uniform electric field E

$\mathbf{F} = (\mathbf{p} \cdot \boldsymbol{\nabla})\mathbf{E}$ $\mathbf{N} = (\mathbf{p} \times \mathbf{E})$ (about the center of the dipole)

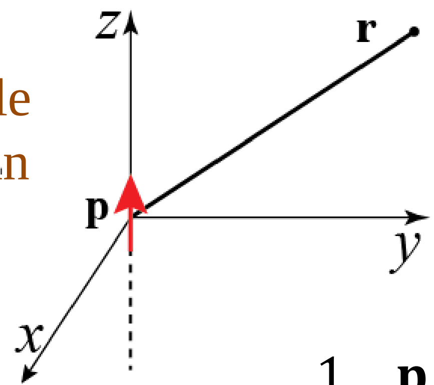
$\mathbf{N} = (\mathbf{p} \times \mathbf{E}) + (\mathbf{r} \times \mathbf{F})$ (about a point **r** away from the center of the dipole)

- Polarization

$\mathbf{P} \equiv$ dipole moment ($\mathbf{p}$) per unit volume

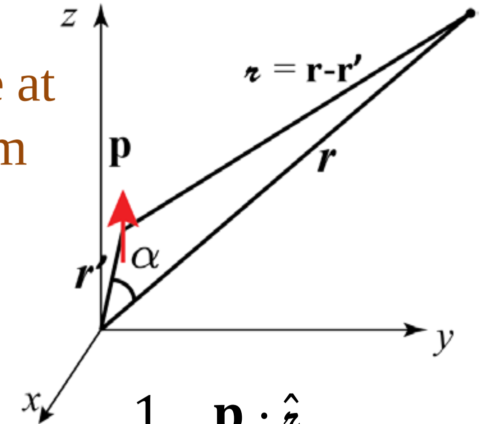
The Field of a Polarized Object:

Point dipole at the origin



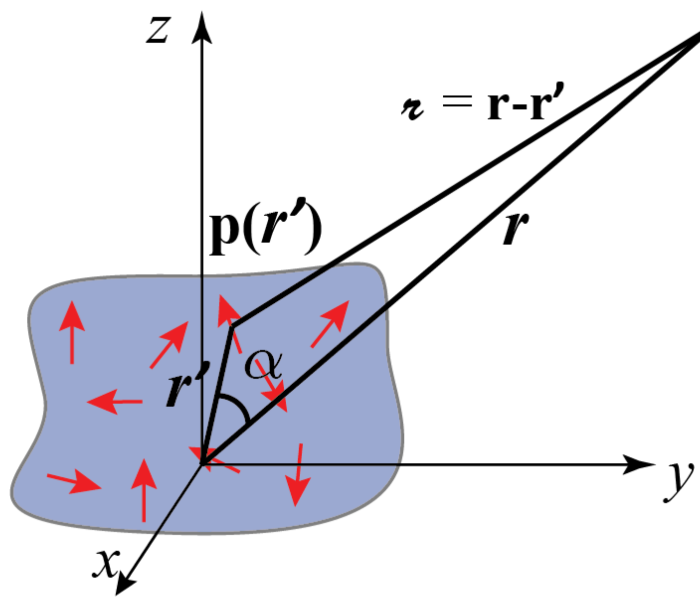
$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

Point dipole at $\mathbf{r}'$ away from the center



$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}$$

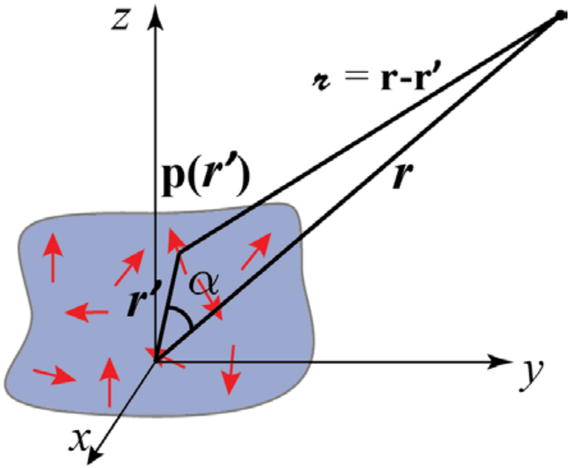
Continuous distribution of dipoles



$$V_{\text{dip}}(\mathbf{r}) = \int_{\mathcal{V}} \frac{1}{4\pi\epsilon_0}\frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

$$\mathbf{p} = \mathbf{P}(\mathbf{r}')\mathbf{d}\boldsymbol{\tau}'$$

The Field of a Polarized Object:



$$V_{\text{dip}}(\mathbf{r}) = \int_{vol} \frac{1}{4\pi\epsilon_0} \frac{\mathbf{P}(\mathbf{r}') \cdot \hat{\mathfrak{r}}}{\mathfrak{r}^2} d\tau' = \int_{vol} \frac{1}{4\pi\epsilon_0} \mathbf{P}(\mathbf{r}') \cdot \left(\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\right) d\tau'$$

$$= \int_{vol} \frac{1}{4\pi\epsilon_0} \mathbf{P}(\mathbf{r}') \cdot \nabla' \left(\frac{1}{\mathfrak{r}}\right) d\tau'$$

[Using $\nabla' \left(\frac{1}{\mathfrak{r}}\right) = \frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}$]

$$= \frac{1}{4\pi\epsilon_0} \int_{vol} \nabla' \cdot \left(\frac{\mathbf{P}(\mathbf{r}')}{\mathfrak{r}}\right) d\tau' - \frac{1}{4\pi\epsilon_0} \int_{vol} \frac{1}{\mathfrak{r}} \nabla' \cdot \mathbf{P}(\mathbf{r}')\, d\tau'$$

[Using $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$]

$$= \frac{1}{4\pi\epsilon_0} \oint_{surf} \frac{\mathbf{P}(\mathbf{r}')}{\mathfrak{r}} \cdot d\mathbf{a}' - \frac{1}{4\pi\epsilon_0} \int_{vol} \frac{1}{\mathfrak{r}} \nabla' \cdot \mathbf{P}(\mathbf{r}')\, d\tau'$$

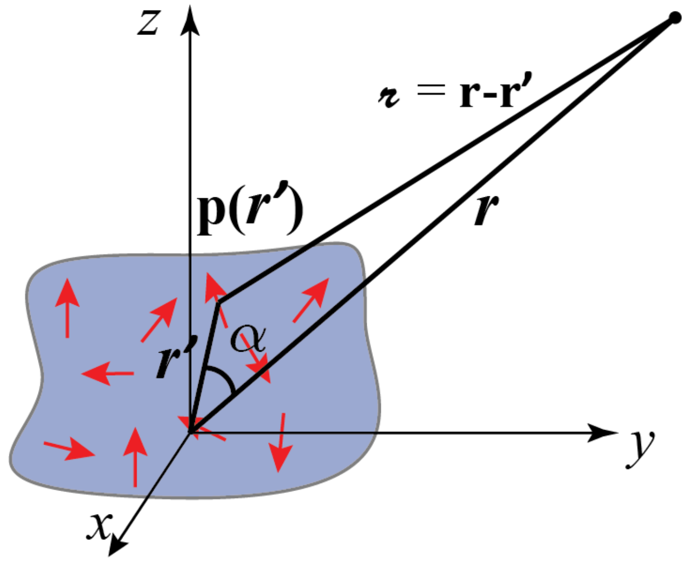
$$= \frac{1}{4\pi\epsilon_0} \oint_{surf} \frac{\mathbf{P}(\mathbf{r}') \cdot \widehat{\mathbf{n}'}}{\mathfrak{r}}\, da' - \frac{1}{4\pi\epsilon_0} \int_{vol} \frac{\nabla' \cdot \mathbf{P}(\mathbf{r}')}{\mathfrak{r}} d\tau'$$

$$\boxed{V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint_{surf} \frac{\sigma_b}{\mathfrak{r}}\, da' + \frac{1}{4\pi\epsilon_0} \int_{vol} \frac{\rho_b}{\mathfrak{r}} d\tau'}$$

$$\sigma_b = \mathbf{P}(\mathbf{r}') \cdot \widehat{\mathbf{n}'}$$

$$\rho_b = -\nabla' \cdot \mathbf{P}(\mathbf{r}')$$

The Field of a Polarized Object:



$$V_{\text{dip}}(\mathbf{r}) = \int_{vol} \frac{1}{4\pi\epsilon_0} \frac{\mathbf{P}(\mathbf{r}') \cdot \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$$

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint_{surf} \frac{\sigma_b}{\mathscr{r}}\, da' + \frac{1}{4\pi\epsilon_0} \int_{vol} \frac{\rho_b}{\mathscr{r}} d\tau'$$

$$\sigma_b = \mathbf{P}(\mathbf{r}') \cdot \widehat{\mathbf{n}}'$$

Surface charge density

$$\rho_b = -\boldsymbol{\nabla}' \cdot \mathbf{P}(\mathbf{r}')$$

Volume charge density

Potential due to a polarized object can be thought of as the sum of the potentials due to a surface charge $\sigma_b = \mathbf{P}(\mathbf{r}') \cdot \widehat{\mathbf{n}}'$ and a volume charge $\rho_b = -\boldsymbol{\nabla}' \cdot \mathbf{P}(\mathbf{r}')$.

Are bound charges real?

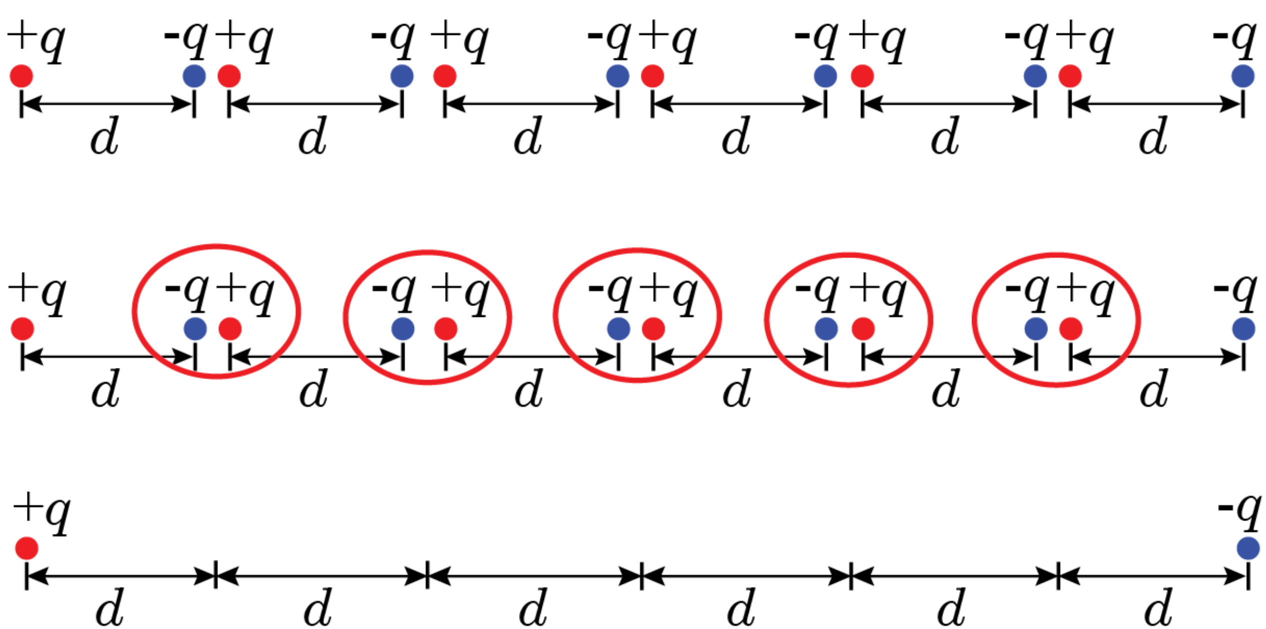
$$\sigma_b = \mathbf{P}(\mathbf{r}') \cdot \widehat{\mathbf{n}'} \qquad \rho_b = -\nabla' \cdot \mathbf{P}(\mathbf{r}')$$

Surface charge density   Volume charge density

These bound charges are not just mathematical constructs. They do appear on the surface and in the volume of the dielectric.

- **Uniform Polarization in one-dimension**

$$\boxed{\sigma_b \neq 0 \qquad \rho_b = 0}$$



Are bound charges real?

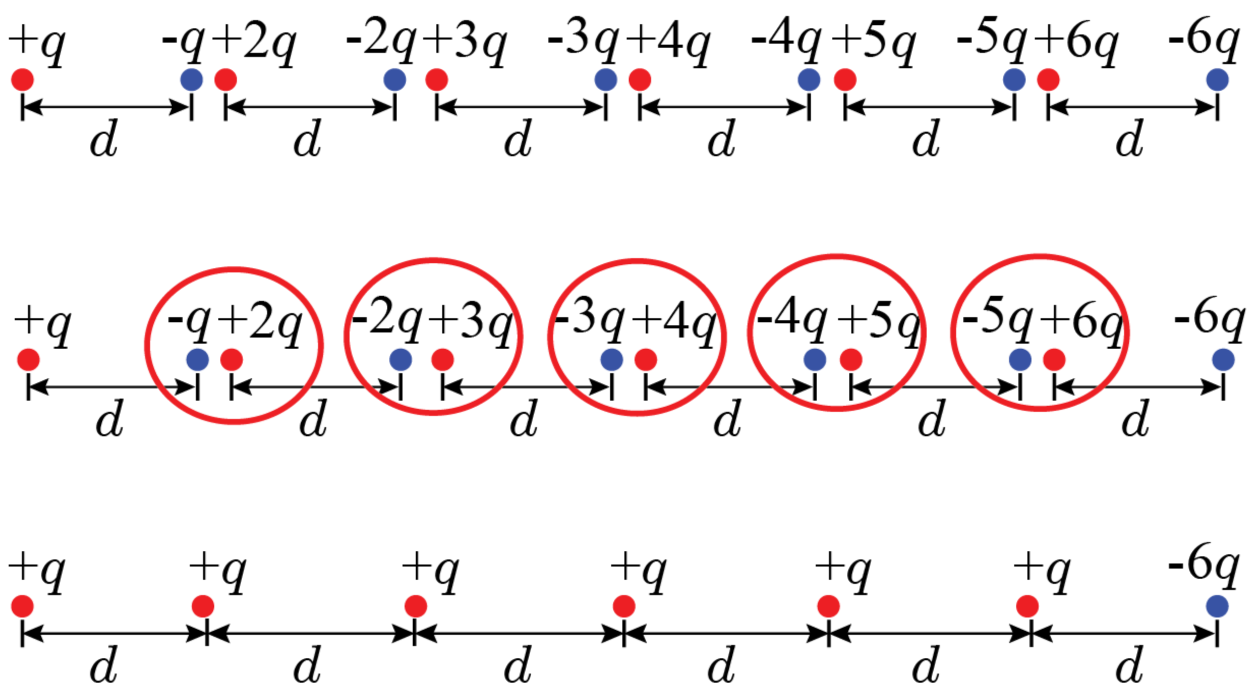
$$\sigma_b = \mathbf{P}(\mathbf{r}') \cdot \widehat{\mathbf{n}'} \qquad\qquad \rho_b = -\boldsymbol{\nabla}' \cdot \mathbf{P}(\mathbf{r}')$$

Surface charge density Volume charge density

These bound charges are not just mathematical constructs. They do appear on the surface and in the volume of the dielectric.

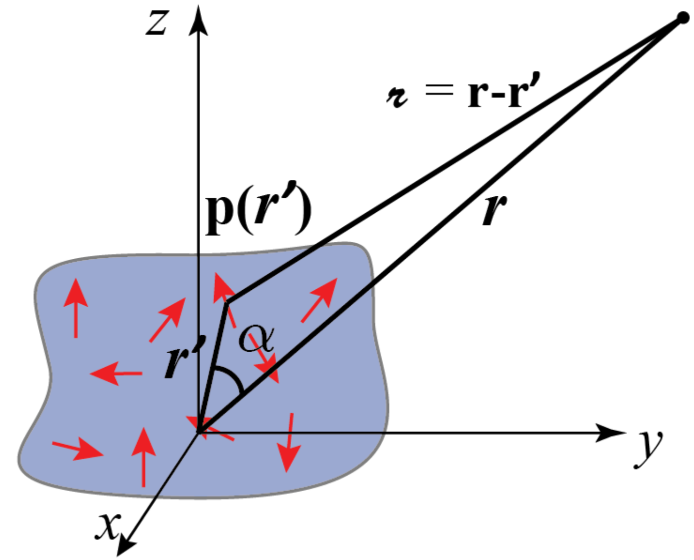
- **Non-Uniform Polarization in one-dimension**

$$\boxed{\sigma_b \neq 0 \qquad \rho_b \neq 0}$$



What is the total bound charge?

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\oint_{surf}\frac{\sigma_b}{\mathscr{r}}\,da' + \frac{1}{4\pi\epsilon_0}\int_{vol}\frac{\rho_b}{\mathscr{r}}\,d\tau'$$



$$\sigma_b = \mathbf{P}(\mathbf{r}')\cdot\widehat{\mathbf{n}}' \qquad\qquad \rho_b = -\nabla'\cdot\mathbf{P}(\mathbf{r}')$$

Prob 4.14 (Griffiths, 3rd Ed.): Prove that the total bound charge is zero.

Total charge $$Q = \oint_{surf}\sigma_b\,da' + \int_{vol}\rho_b\,d\tau'$$

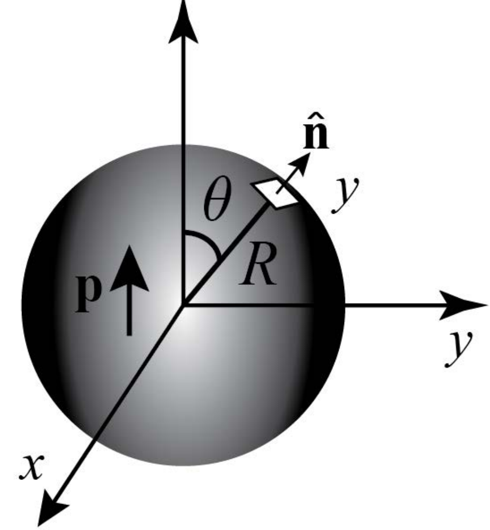
$$= \oint_{surf}\mathbf{P}(\mathbf{r}')\cdot\widehat{\mathbf{n}}'da' - \int_{vol}\nabla'\cdot\mathbf{P}(\mathbf{r}')d\tau'$$

$$= \oint_{surf}\mathbf{P}(\mathbf{r}')\cdot d\mathbf{a}' - \int_{vol}\nabla'\cdot\mathbf{P}(\mathbf{r}')d\tau'$$

$$= \int_{vol}\nabla'\cdot\mathbf{P}(\mathbf{r}')d\tau' - \int_{vol}\nabla'\cdot\mathbf{P}(\mathbf{r}')d\tau' = 0$$

The Field of a Polarized Object:

Ex. 4.2 (Griffiths, 3rd Ed.): Find the electric field of a sphere of radius R, if it is uniformly polarized $\mathbf{P}(\mathbf{r}') = P\,\hat{\mathbf{z}}$.



$$\rho_b = -\nabla' \cdot \mathbf{P}(\mathbf{r}') = -\nabla' \cdot (P\,\hat{\mathbf{z}}) \;= 0$$

$$\sigma_b = \mathbf{P}(\mathbf{r}') \cdot \widehat{\mathbf{n}'} \quad = P\cos\theta$$

The potential due to a uniformly polarized sphere is equal to the potential due to a spherical surface charge density $\sigma_b = P\cos\theta$

For this charge distribution, the potential is calculated in (Ex. 3.9 Griffiths, 3rd Ed.)

For $r \geq R$

$$V(r,\theta) = \frac{P}{3\epsilon_0}\frac{R^3}{r^2}\cos\theta = \frac{R^3}{3\epsilon_0}\frac{\mathbf{P}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{3\epsilon_0}\frac{\left(\frac{4\pi}{3}\right)R^3}{\left(\frac{4\pi}{3}\right)}\frac{\mathbf{P}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

$$\boxed{\mathbf{p} = \left(\frac{4\pi}{3}\right)R^3\mathbf{P}}$$

total dipole moment of the sphere

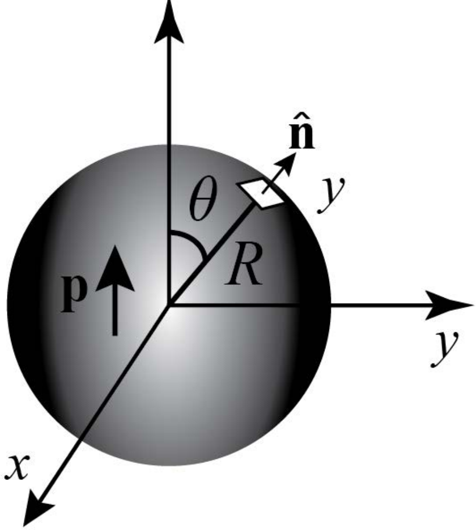
$$\Rightarrow \quad \mathbf{E} = -\nabla V = \mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta})$$

For $r \leq R$

$$V(r,\theta) = \frac{P}{3\epsilon_0}r\cos\theta \Rightarrow \mathbf{E} = -\nabla V \;= -\frac{P}{3\epsilon_0}\hat{\mathbf{z}} = -\frac{p}{4\pi\epsilon_0 R^3}\hat{\mathbf{z}}$$

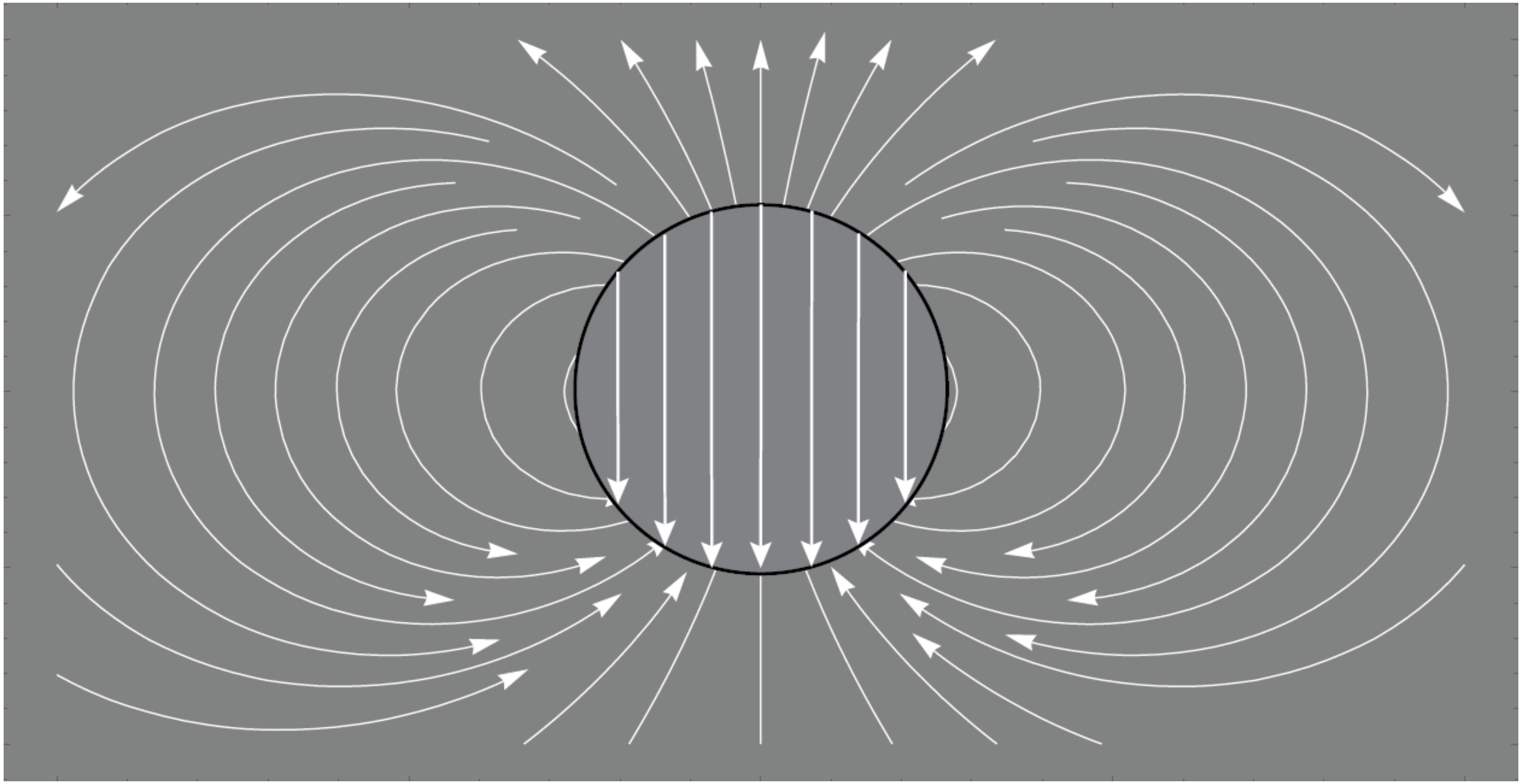
The Field of a Polarized Object:

Ex. 4.2 (Griffiths, 3rd Ed.): Find the electric field of a sphere of radius R, if it is uniformly polarized $\mathbf{P}(\mathbf{r}') = P\,\hat{\mathbf{z}}$.



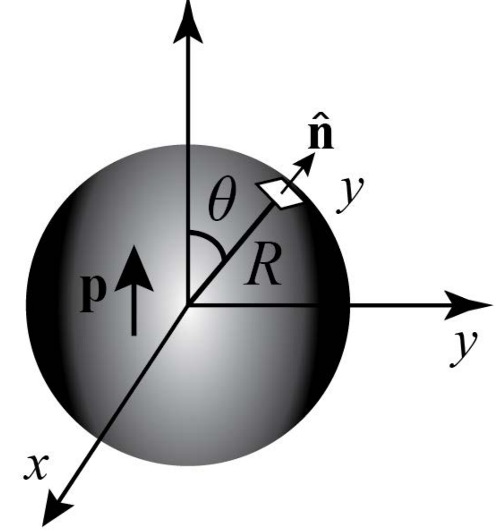
For $r \geq R$ $\quad \mathbf{E} = \dfrac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta})$

For $r \leq R$ $\quad \mathbf{E} = -\dfrac{p}{4\pi\epsilon_0 R^3}\hat{\mathbf{z}}$



The Field of a Polarized Object:

Prob 4.10 (Griffiths, 3rd Ed.): Find the bound charges of a sphere of radius R, if its polarization is $\mathbf{P}(\mathbf{r}') = k\mathbf{r}$.



Volume charge

$$\rho_b = -\nabla' \cdot \mathbf{P}(\mathbf{r}')$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 kr) \quad = -\frac{1}{r^2}3kr^2 \quad = -3k$$

Surface charge

$$\sigma_b = \mathbf{P}(\mathbf{r}') \cdot \widehat{\mathbf{n}'} \qquad = kR$$

Q: What is the electric field outside the sphere?

Volume and surface charge distributions are both symmetric with respect to the center of the sphere. So, the total charge can be thought of as being concentrated at the center

$$\text{Total charge } Q = \oint_{surf} \sigma_b \, da' + \int_{vol} \rho_b \, d\tau'$$

$$= kR \times 4\pi R^2 \quad +(-3k) \times \frac{4\pi}{3} R^3$$

$$= 0$$

So the electric field outside the sphere is zero.