Semester II, 2017-18
Department of Physics, IIT Kanpur

PHY103A: Lecture # 15

(Text Book: Intro to Electrodynamics by Griffiths, 3rd Ed.)

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Notes

- Quiz # 1 on Tuesday (06-Feb-2018), during tutorial
- Quiz timing: 11:00 – 11:15 am (15 Min).
- Course coverage (up to HW # 4, Griffiths Ch 3)
- Need to remember the basic formulas.

Summary of Lecture # 14:

- Gauss's law in the presence of Dielectrics

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$$ **Electric displacement**

$$\nabla \cdot \mathbf{D} = \rho_f \longleftrightarrow \oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{enc}}$$

- Boundary Conditions on electric displacement

$$D^{\perp}{}_{\text{above}} - D^{\perp}{}_{\text{below}} = \sigma_f$$

$$D^{||}{}_{\text{above}} - D^{||}{}_{\text{below}} = P^{||}{}_{\text{above}} - P^{||}{}_{\text{below}}$$

Questions 1:

$$\epsilon_0 \, \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f \qquad \nabla \cdot \mathbf{D} = \rho_f \longleftrightarrow \oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{enc}}$$

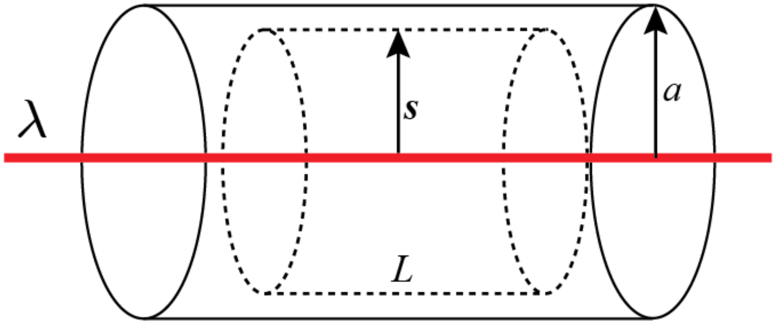
Why did we not include the surface charge σ_b in the derivation of the Gauss's law?

We don't need to. Volume charge takes care of everything.

When the Gaussian surface is drawn in the bulk of the material, the surface charge does not come into picture.

On the surface, the Gauss's law anyway cannot be applied.

$$\mathbf{D} = \frac{\lambda}{2\pi s}\hat{\mathbf{s}}$$



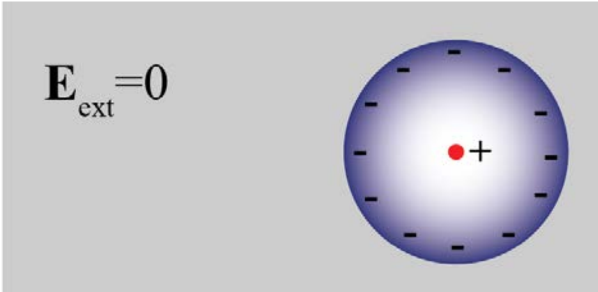
For $(s > a)$ $\quad \mathbf{E} = \dfrac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}$

(since $\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$ and $\mathbf{P} = 0$)

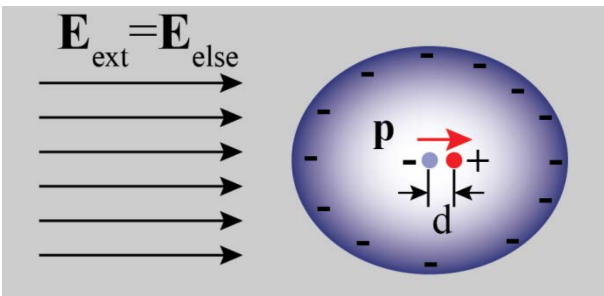
For $s < a$, $\mathbf{E}$ cannot be found since $\mathbf{P}$ for $s < a$ is not known,.

Linear Dielectrics

- Two ways that an atom/molecule acquires dipole moment



Stretch of an atom/molecule



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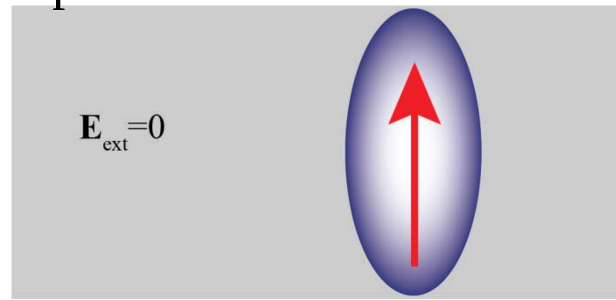
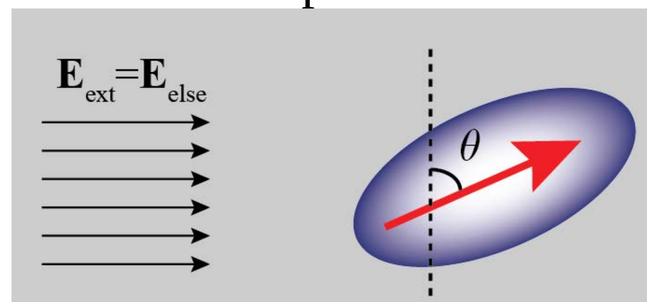


Rotation of a polar atom/molecule



$\Rightarrow$ $\mathbf{p} = \boldsymbol{\alpha}\, \mathbf{E}_{\text{else}}$

- $\boldsymbol{\alpha}$ is the atomic polarizability
- $\mathbf{E}_{\text{else}}$ is the external field, caused by everything except the atom

- Dipole moment $\mathbf{p}$ is a microscopic quantity. However, polarization $\mathbf{P}$ (dipole moment per unit volume) is a macroscopic quantity.

If $\mathbf{E}$ is not too strong

$$\mathbf{P} = \epsilon_0 \chi_e\, \mathbf{E}$$

- χ_e is called the electrical susceptibility and depends on the details (microscopic and macroscopic of the medium)
- $\mathbf{E}$ is the total field caused by everything.
- How are $\boldsymbol{\alpha}$ and χ_e connected?

Linear Dielectrics

$\mathbf{P} = \epsilon_0 \chi_e \, \mathbf{E}$

- Medium that obeys this equation is called the linear dielectric
- When polarization is proportional to the square or higher-order terms in **E**, the medium is called the nonlinear dielectric.
- At strong enough **E** every medium becomes nonlinear.
- In general χ_e is a tensor and is called the susceptibility tensor.

So, what about the electric displacement **D** for a linear dielectric?

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \;\; = \epsilon_0 \, \mathbf{E} + \epsilon_0 \chi_e \, \mathbf{E} \;\; = \epsilon_0 \, (1 + \chi_e)\mathbf{E}$$

Define: $\epsilon \equiv \epsilon_0 (1 + \chi_e)$ Define: $\epsilon_r \equiv \frac{\epsilon}{\epsilon_0} = (1 + \chi_e)$

$$\boxed{\mathbf{D} = \epsilon \mathbf{E}}$$

- The electric displacement **D** is also proportional to **E**
- ϵ is called the permittivity of the material.
- ϵ_r is called the relative permittivity or the dielectric constant.
- $n = \sqrt{\epsilon_r}$ is called the refractive index.

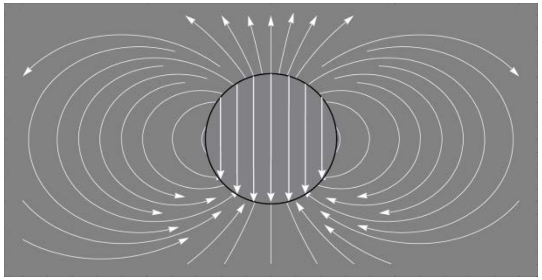
How are α and χ_e connected?

Dipole moment $\mathbf{p} = \alpha\, \mathbf{E}_{\text{else}}$

Suppose the density of atom (no. of atoms per unit volume) is $N = \frac{1}{\left(\frac{4\pi}{3}\right)R^3}$

So, Polarization (dipole moment per unit volume) is $\mathbf{P} = N\boldsymbol{\alpha}\, \mathbf{E}_{\text{else}}$

But $\mathbf{P} = \epsilon_0 \chi_e\, \mathbf{E}$ Where $\mathbf{E} = \mathbf{E}_{\text{else}} + \mathbf{E}_{\text{self}}$



$$\mathbf{E}_{\text{self}} = -\frac{p}{4\pi\epsilon_0 R^3}\hat{\mathbf{z}} = -\frac{\mathbf{p}}{4\pi\epsilon_0 R^3} = -\frac{\alpha}{4\pi\epsilon_0 R^3}\mathbf{E}_{\text{else}}$$

$$\mathbf{E} = \mathbf{E}_{\text{else}} + \mathbf{E}_{\text{self}} = \mathbf{E}_{\text{else}} - \frac{\alpha}{4\pi\epsilon_0 R^3}\mathbf{E}_{\text{else}} = \mathbf{E}_{\text{else}}\left(1 - \frac{\alpha}{4\pi\epsilon_0 R^3}\right) = \mathbf{E}_{\text{else}}\left(1 - \frac{N\alpha}{3\epsilon_0}\right)$$

Now, $\mathbf{P} = \epsilon_0 \chi_e\, \mathbf{E}$

$$\Rightarrow \quad N\alpha\, \mathbf{E}_{\text{else}} = \epsilon_0 \chi_e \mathbf{E}_{\text{else}}\left(1 - \frac{N\alpha}{3\epsilon_0}\right) \quad \Rightarrow \quad \alpha = \frac{3\epsilon_0}{N}\frac{\chi_e}{(3+\chi_e)}$$

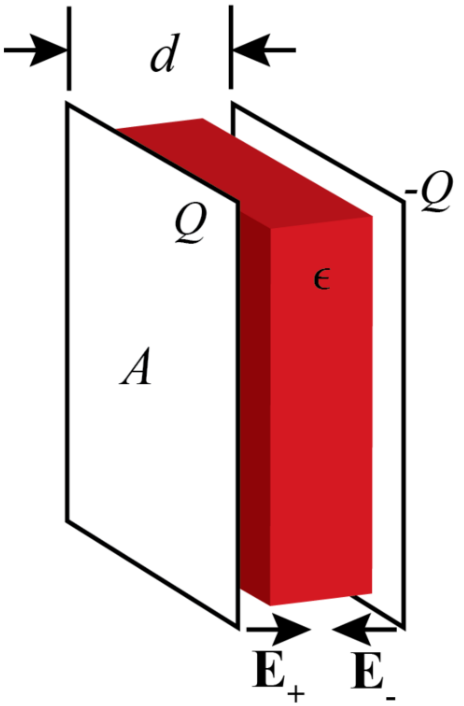
Using $\epsilon_r = (1+\chi_e)$

$$\alpha = \frac{3\epsilon_0}{N}\frac{(\epsilon_r - 1)}{(\epsilon_r + 2)}$$

- Clausius-Mossotti formula
- Connects a microscopic quantity to a macroscopic quantity

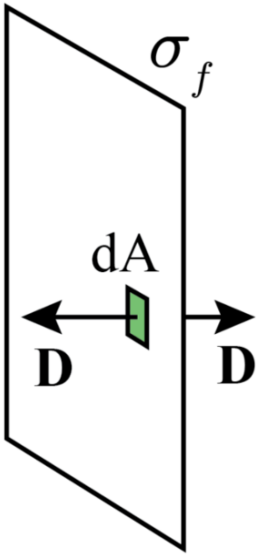
Capacitor with a dielectric filling

Q: For a parallel plate capacitor, what is the capacitance if the space between the plates is filled with a material of dielectric constant ϵ_r. (The vacuum capacitance is $C_{\mathrm{vac}} = \frac{A\epsilon_0}{d}$)



Boundary condition on electric displacement is

$$D^{\perp}{}_{\mathrm{above}} - D^{\perp}{}_{\mathrm{below}} = \sigma_f$$



This gives $D - (-D) = \sigma_f$

Or, $D = \dfrac{\sigma_f}{2}$ Thus, $E = \dfrac{\sigma_f}{2\epsilon}$

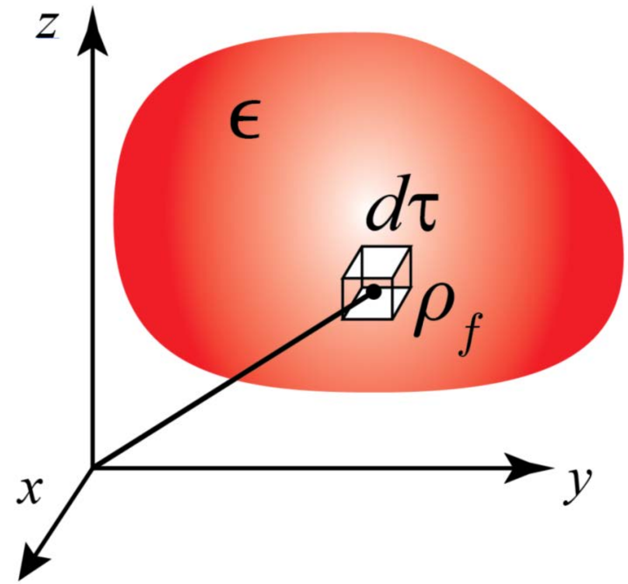
The electric field between the parallel plates

$$\mathbf{E} = \mathbf{E}_+ - \mathbf{E}_- = \frac{\sigma_f}{2\epsilon} - \frac{(-\sigma_f)}{2\epsilon} = \frac{\sigma_f}{\epsilon} = \frac{Q}{A\epsilon}$$

The potential difference $V = -\int \mathbf{E}.d\mathbf{l} = E\,d = \dfrac{Q}{A\epsilon}d$

Capacitance $C = \dfrac{Q}{V} = \dfrac{A\epsilon}{d} = \dfrac{A\epsilon_0}{d}\dfrac{\epsilon}{\epsilon_0} = C_{\mathrm{vac}}\epsilon_r$

Energy in Dielectric Systems



With no polarization, the energy stored in an electrostatic system is

$$W = \frac{\epsilon_0}{2}\int_{all\ space} E^2 d\tau$$

With polarization, what is the energy stored in an electrostatic system?

Work done to move a charge (free) Δq_f is

$$\Delta W = \Delta q_f V = \int \Delta\rho_f V d\tau$$

($\Delta\rho_f$ is the change in the free charge density)

Gauss's Law: $\rho_f = \nabla \cdot \mathbf{D}$ $\Rightarrow$ $\Delta\rho_f = \Delta(\nabla \cdot \mathbf{D}) = \nabla \cdot (\Delta\mathbf{D})$

$$\Delta W = \int [\nabla \cdot (\Delta\mathbf{D})] V d\tau$$

$$\Delta W = \int \nabla \cdot (V\Delta\mathbf{D}) d\tau - \int \Delta\mathbf{D} \cdot \nabla V\, d\tau$$

(Using the product rule $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$)

$$\Delta W = \int \nabla \cdot (V\Delta\mathbf{D}) d\tau + \int \Delta\mathbf{D} \cdot \mathbf{E}\, d\tau$$

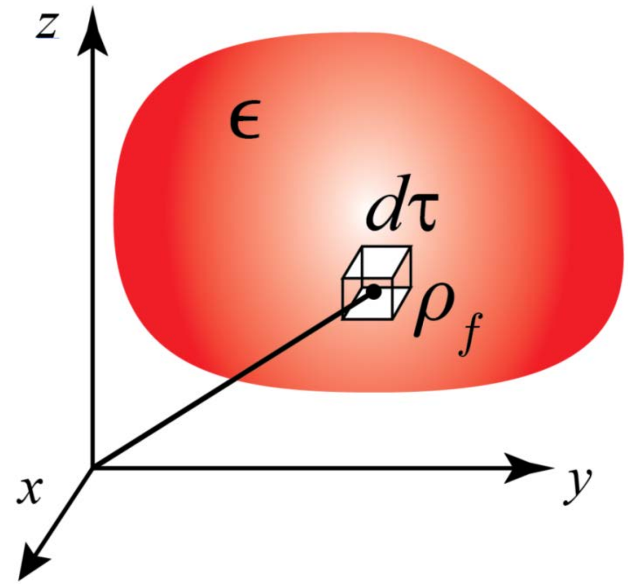
(Using $-\nabla V = \mathbf{E}$)

Energy in Dielectric Systems

With no polarization, the energy stored in an electrostatic system is

$$W = \frac{\epsilon_0}{2} \int_{all\ space} E^2 d\tau$$

With polarization, what is the energy stored in an electrostatic system?



$$\Delta W = \int_{vol} \nabla \cdot (V\Delta\mathbf{D}) d\tau + \int_{vol} \Delta\mathbf{D} \cdot \mathbf{E}\, d\tau$$

$$\Delta W = \oint_{surf} V\Delta\mathbf{D} \cdot d\mathbf{a} + \int_{vol} \Delta\mathbf{D} \cdot \mathbf{E}\, d\tau$$

Using the divergence theorem $\int_{Vol} (\nabla \cdot \mathbf{A}) d\tau = \oint_{Surf} \mathbf{A} \cdot d\mathbf{a}$

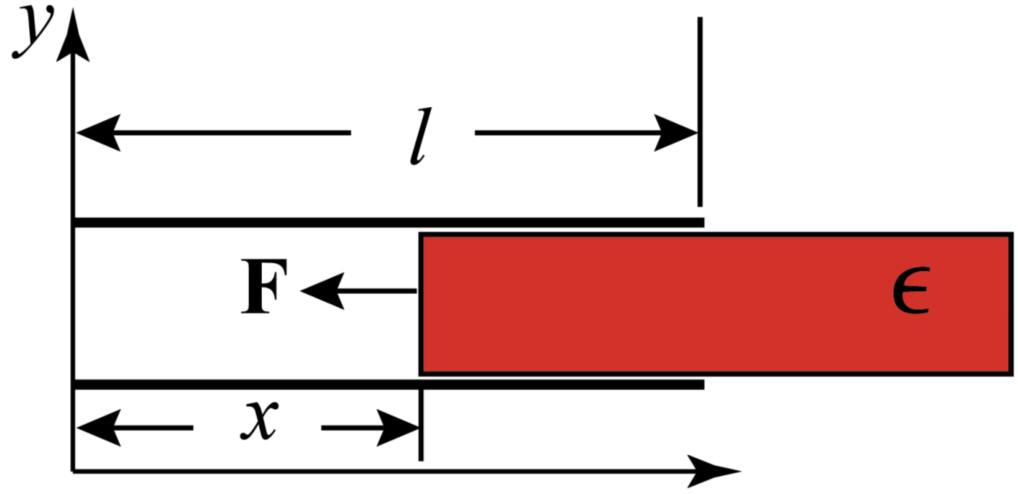
$$\Delta W = \int_{all\ space} \Delta\mathbf{D} \cdot \mathbf{E}\, d\tau$$

For a linear dielectric $\mathbf{D} = \epsilon\mathbf{E}$ $\Rightarrow$ $\frac{1}{2}\Delta(\mathbf{D} \cdot \mathbf{E}) = \frac{1}{2}\Delta(\epsilon E^2) = \epsilon(\Delta\mathbf{E}) \cdot \mathbf{E} = (\Delta\mathbf{D}) \cdot \mathbf{E}$

$$\Delta W = \frac{1}{2} \int_{all\ space} \Delta(\mathbf{D} \cdot \mathbf{E})\, d\tau = \Delta\left(\frac{1}{2} \int_{all\ space} \mathbf{D} \cdot \mathbf{E}\, d\tau\right)$$

$$\boxed{W = \frac{1}{2} \int_{all\ space} \mathbf{D} \cdot \mathbf{E}\, d\tau}$$

Forces on Dielectrics



Force on the dielectric is

$$F = -\frac{dW}{dx}$$

Energy stored in the capacitor is

$$W = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}$$

So, $F = -\frac{dW}{dx} = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx}$ Thus, $\boxed{F = \frac{1}{2}V^2\frac{dC}{dx}}$ **Force on the dielectric**

But $C = \frac{\epsilon_0 a\, x}{d} + \epsilon_r\frac{\epsilon_0 a\,(l-x)}{d} = \frac{\epsilon_0 a}{d}[x + \epsilon_r(l-x)] = \frac{\epsilon_0 a}{d}[\epsilon_r l - (\epsilon_r - 1)x]$

$$= \frac{\epsilon_0 a}{d}[\epsilon_r l - \chi_e x]$$

So, $\frac{dC}{dx} = -\frac{\epsilon_0 \chi_e a}{d}$

Therefore, $F = -\frac{\epsilon_0 \chi_e a}{2d}V^2$ Force is in the negative x direction