Semester II, 2017-18
Department of Physics, IIT Kanpur

PHY103A: Lecture # 17

(Text Book: Intro to Electrodynamics by Griffiths, 3rd Ed.)

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07-Feb-2018

Notes

- Homework # 6 has been posted on the course webpage
- Solutions to HW # 5 have also been posted.
- Quiz # 1 is on Tuesday (Feb 13); 11:00 – 11:15 am.

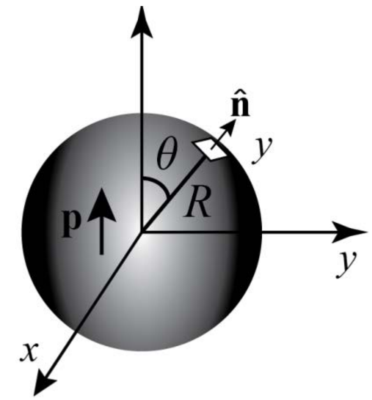
Conceptual Clarification (Multipole expansion versus dipole field)

Multipole Expansion:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\mathbf{r}')d\tau' + \frac{1}{r^2}\int r'(\cos\alpha)\rho(\mathbf{r}')d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')d\tau' + \cdots\right]$$

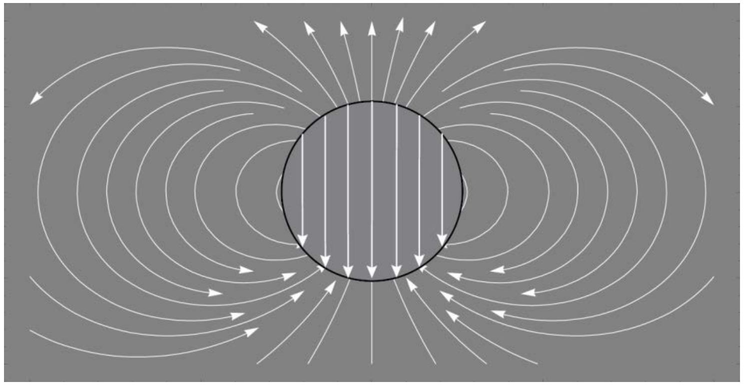
- It is an exact expression but **at large *r*** the expression can be approximated by just the first non-zero term in the expansion.

The Field of a Polarized Object: Ex. 4.2 (Griffiths, 3rd Ed.):



For $r \geq R$ $\quad \mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta})$

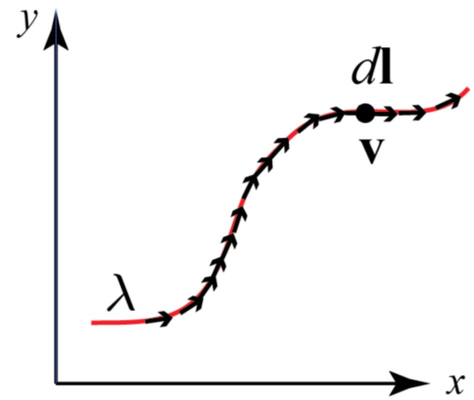
For $r \leq R$ $\quad \mathbf{E} = -\frac{p}{4\pi\epsilon_0 R^3}\hat{\mathbf{z}}$



- The dipole field can be calculated **at all *r***

Summary of Lecture # 16:

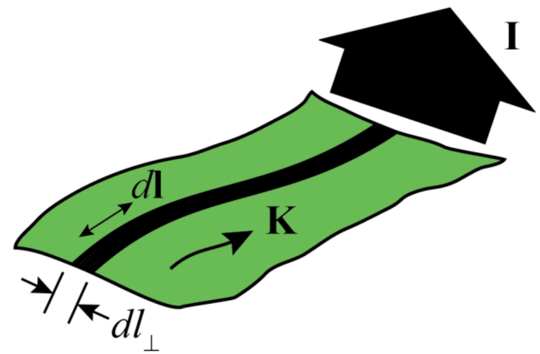
- Magnetic Force: $\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B})$ **Lorentz Force Law**



- The work done by a magnetic force is zero !

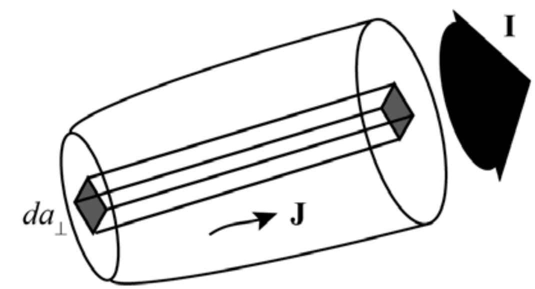
- Current $\quad \mathbf{I} = \dfrac{\lambda\, d\mathbf{l}}{dt} = \lambda \mathbf{v}$

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{I} \times \mathbf{B})\, dl = I \int (d\mathbf{l} \times \mathbf{B})$$



- Surface Current Density $\quad \mathbf{K} = \dfrac{d\mathbf{I}}{dl_\perp} = \sigma \mathbf{v}$

$$\mathbf{F}_{\text{mag}} = (\mathbf{v} \times \mathbf{B})Q = \int (\mathbf{K} \times \mathbf{B})\, da$$

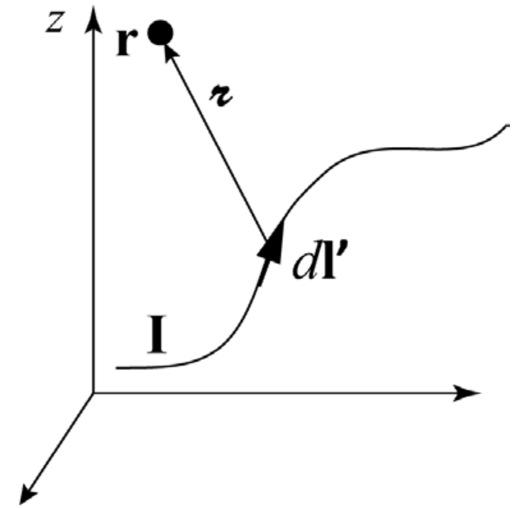


- Volume Current Density $\quad \mathbf{J} = \dfrac{d\mathbf{I}}{da_\perp} = \rho \mathbf{v}$

$$\mathbf{F}_{\text{mag}} = (\mathbf{v} \times \mathbf{B})Q = \int (\mathbf{J} \times \mathbf{B})\, d\tau$$

- The Continuity Equation $\quad \nabla \cdot \mathbf{J} = -\dfrac{d\rho}{dt}$

The Biot-Savart Law



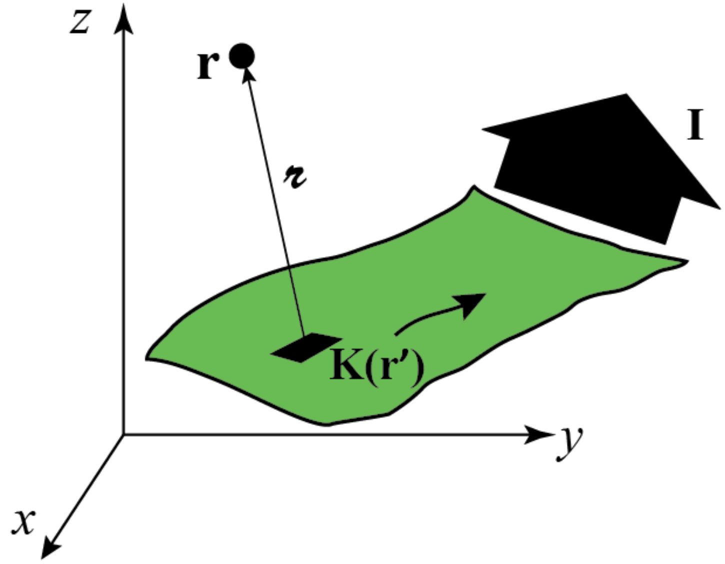
The magnetic field produced by a steady line current

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

- μ_0 is the permeability of free space
- $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$
- The unit of magnetic field is Newton per Ampere-meter, or Tesla
- 1 Tesla is a very strong magnetic field. Earth’s magnetic field is about 10^{-4} times smaller
- Biot-Savart law for magnetic field is analogous to Coulomb’s law for electric field

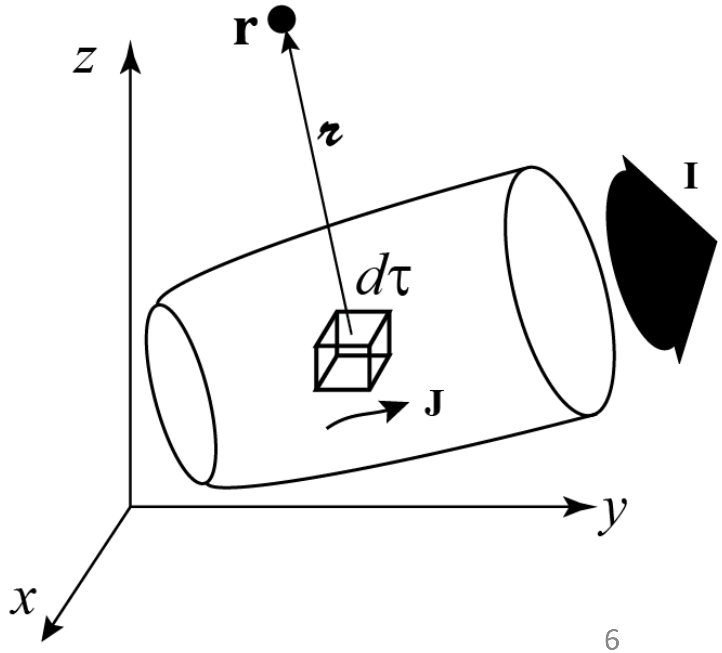
The Biot-Savart Law

The magnetic field produced by a surface current



$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} da'$$

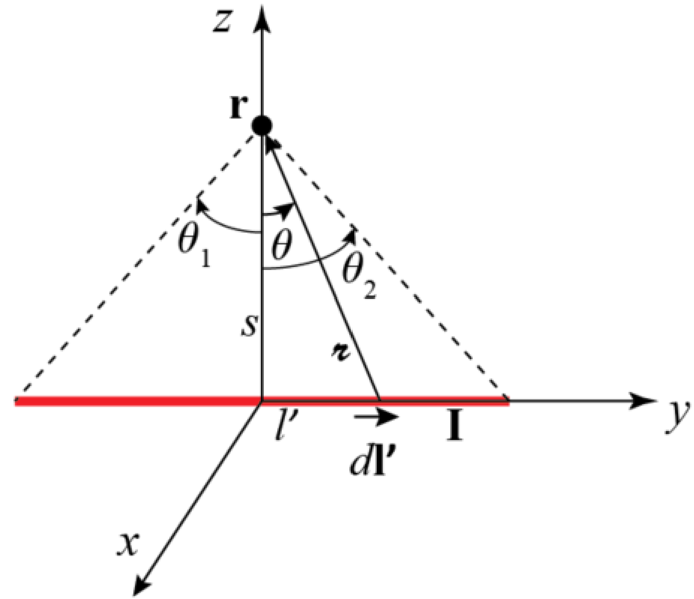
The magnetic field produced by a volume current



$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} d\tau'$$

The Biot-Savart Law

Ex. 5.5 (Griffiths, 3rd Ed.): Calculate the magnetic field due to a long straight wire carrying a steady current I .



$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\mathbf{B}(\mathbf{r}) = B\,\hat{\mathbf{x}}$$

$$|d\mathbf{l}' \times \hat{\mathscr{r}}| = dl' \cos\theta$$

$$l' = s \tan\theta \quad \Rightarrow \quad dl' = \frac{s}{\cos^2\theta} d\theta$$

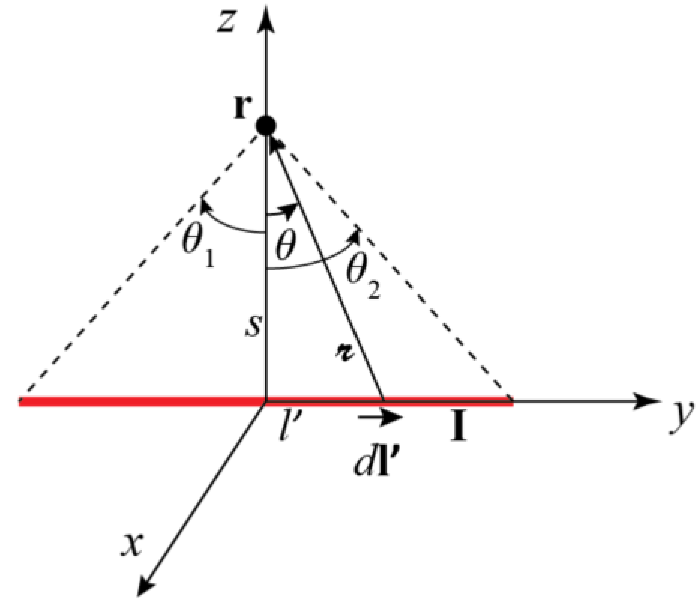
$$s = \mathscr{r} \cos\theta \quad \Rightarrow \quad \frac{1}{\mathscr{r}^2} = \frac{\cos^2\theta}{s^2}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int_{\theta_1}^{\theta_2} \left(\frac{\cos^2\theta}{s^2}\right)\left(\frac{s}{\cos^2\theta}\right) \cos\theta \, d\theta \, \hat{\mathbf{x}} = \frac{\mu_0 I}{4\pi s} \int_{-\theta_1}^{\theta_2} \cos\theta \, d\theta \, \hat{\mathbf{x}}$$

$$= \frac{\mu_0 I}{4\pi s} (\sin\theta_2 + \sin\theta_1)\hat{\mathbf{x}}$$

The Biot-Savart Law

Ex. 5.5 (Griffiths, 3rd Ed.): Calculate the magnetic field due to a long straight wire carrying a steady current I .



$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

$$= \frac{\mu_0 I}{4\pi s} (\sin\theta_2 + \sin\theta_1)\hat{\mathbf{x}}$$

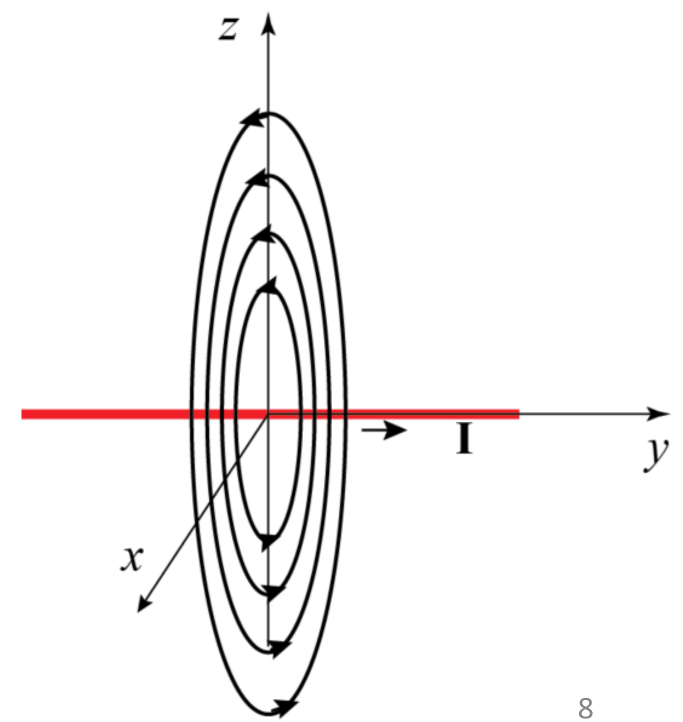
Field due to an infinite wire ?

$$\theta_1 = \frac{\pi}{2} \qquad \theta_2 = \frac{\pi}{2}$$

$$\mathbf{B}(\mathbf{r}) \quad = \frac{\mu_0 I}{4\pi s} (\sin\theta_2 + \sin\theta_1)\hat{\mathbf{x}}$$

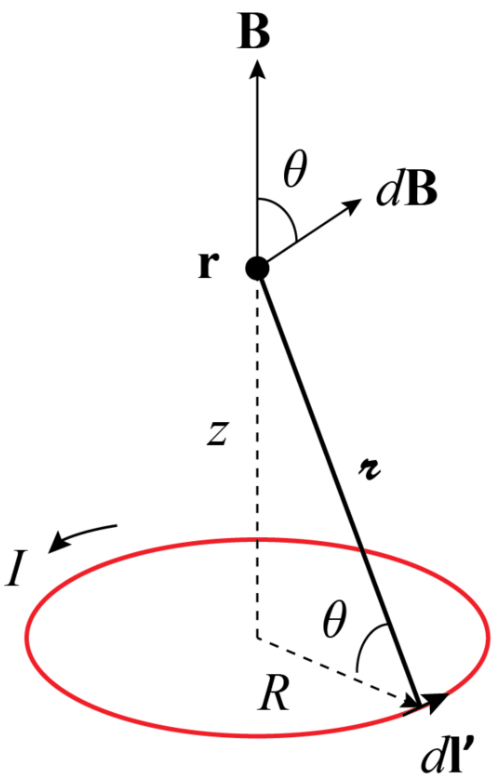
$$= \frac{\mu_0 I}{4\pi s} (1 + 1)\hat{\mathbf{x}}$$

$$= \frac{\mu_0 I}{2\pi s} \hat{\mathbf{x}}$$



The Biot-Savart Law

Ex. 5.6 (Griffiths, 3rd Ed.): Find the magnetic field **B**(**r**) field above the center of a loop (radius R, current I).



$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

The line element $d\mathbf{l}'$ produces the field $d\mathbf{B}$ at $\mathbf{r}$. The horizontal components of this field cancels out.

The vertical component of this field is

$$dB(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \frac{dl'}{\mathscr{r}^2} \cos\theta$$

The total field, which is in the z direction, is

$$\mathbf{B}(z) = \frac{\mu_0 I}{4\pi} \int \frac{\cos\theta}{\mathscr{r}^2} dl' = \frac{\mu_0 I}{4\pi} \frac{\cos\theta}{\mathscr{r}^2} \int dl' = \frac{\mu_0 I}{4\pi} \frac{R}{\mathscr{r}^3} (2\pi R)$$

$$= \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}}$$

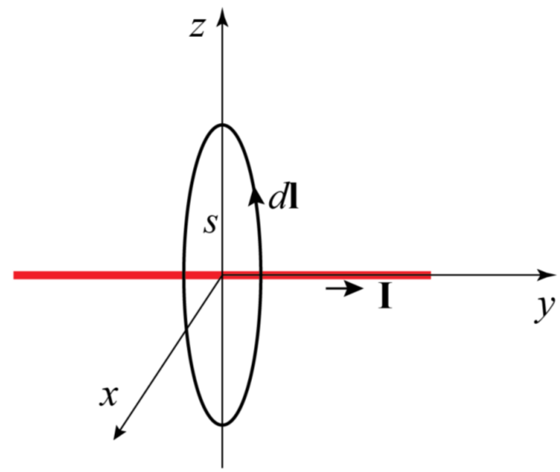
What is the field at the center?

$$\mathbf{B}(0) = \frac{\mu_0 I}{2} \frac{R^2}{R^3} = \frac{\mu_0 I}{2R}$$

The Divergence and Curl of B

What is the divergence of **B** ? $\nabla \cdot \mathbf{B} = 0$

What is the curl of **B** ? Should be $\nabla \times \mathbf{B} \neq \mathbf{0}$



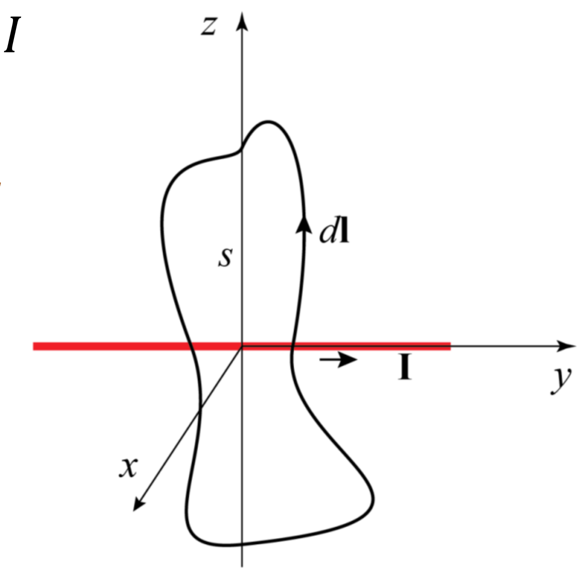
Check for the case of straight wire with current I

$$\mathrm{B}(s) = \frac{\mu_0 I}{2\pi s}$$

For circular path of radius s

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \frac{\mu_0 I}{2\pi s} \oint dl = \frac{\mu_0 I}{2\pi s} 2\pi s = \mu_0 I$$

The line integral is independent of *s*



For an arbitrary path enclosing the current carrying wire

The field is best represented in the cylindrical coordinate

$$\mathbf{B}(s) = \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}} \qquad d\mathbf{l} = ds\, \hat{\mathbf{s}} + s d\phi\, \hat{\boldsymbol{\phi}} + dz\, \hat{\mathbf{z}}$$

So, $$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}} \cdot (ds\, \hat{\mathbf{s}} + s d\phi\, \hat{\boldsymbol{\phi}} + dz\, \hat{\mathbf{z}}) = \frac{\mu_0 I}{2\pi} \int_0^{2\pi} d\phi = \mu_0 I$$

The Divergence and Curl of B

What is the divergence of **B** ? $\nabla \cdot \mathbf{B} = 0$

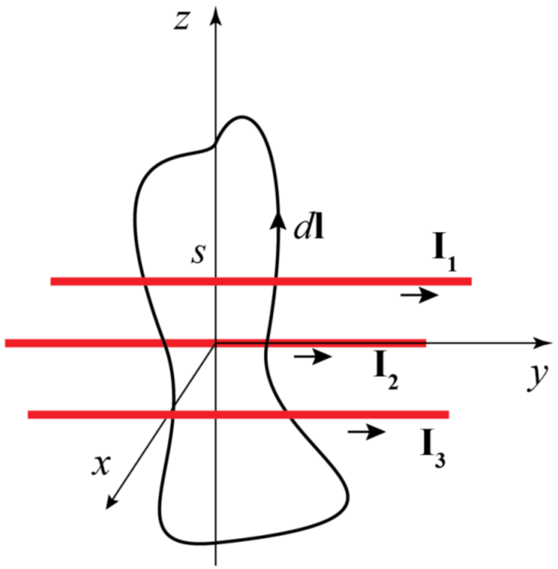
What is the curl of **B** ? Should be $\nabla \times \mathbf{B} \neq \mathbf{0}$

Check for the case of straight wire with current *I*

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I$$

If the path encloses more than one current carrying wire

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_1 + \mu_0 I_2 + \mu_0 I_3 = \mu_0 I_{\text{enc}}$$



But $I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a}$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a} \Rightarrow \boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}}$$

- Is this valid only for straight wires? No
- It is valid in general

The Ampere's Law

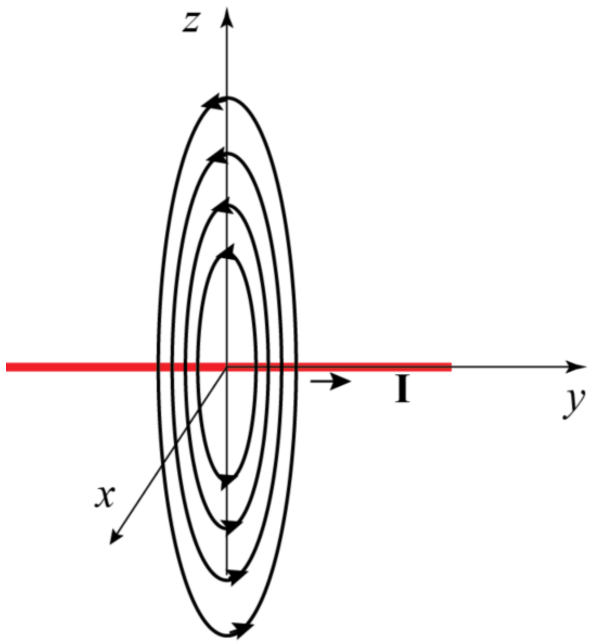
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ Ampere's law in differential form

$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$ Ampere's law in integral form

- Ampere's law is analogous to Gauss's law
- Ampere's law makes the calculation of magnetic field very easy if there is symmetry.
- If there is no symmetry, one has to use Biot-Savart law to calculate the magnetic field.

The Ampere's Law

Ex. 5.5 (Griffiths, 3rd Ed.): Calculate the magnetic field due to an infinitely long straight wire carrying a steady current I .

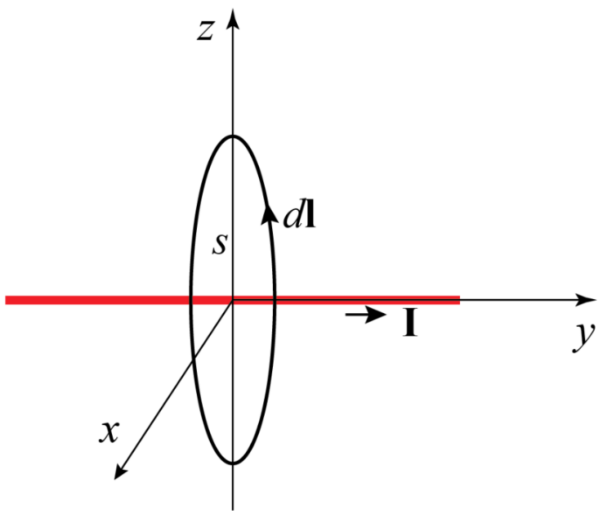


Using Biot-Savart Law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{2\pi s}\hat{\mathbf{x}}$$

We want to do this problem using Ampere's law

Make an Amperian loop of radius s enclosing the current



Since it is an infinite wire, the magnetic field must be circularly symmetric. Therefore,

$$\oint \mathbf{B} \cdot d\mathbf{l} = B\oint dl = B2\pi s = \mu_0 I_{\text{enc}} = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi s}$$

So Easy !!