Semester II, 2017-18

Department of Physics, IIT Kanpur

PHY103A: Lecture # 18

(Text Book: Intro to Electrodynamics by Griffiths, 3rd Ed.)

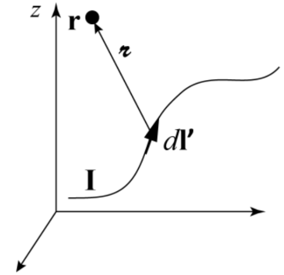
Anand Kumar Jha

09-Feb-2018

Summary of Lecture # 17:

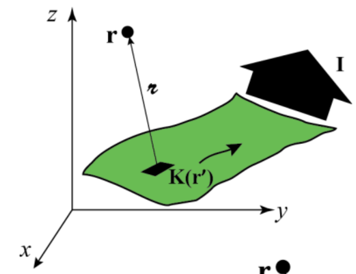
- The magnetic field produced by a steady line current

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,dl'$$



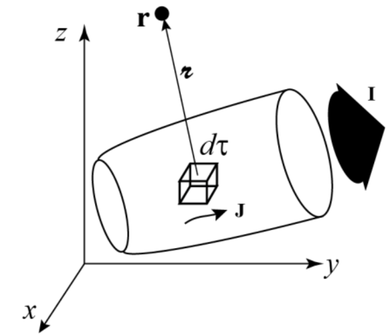
- The magnetic field produced by a surface current

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,da'$$



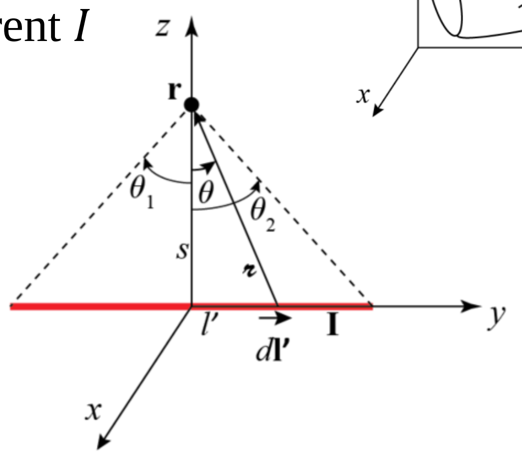
- The magnetic field produced by a volume current

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$$



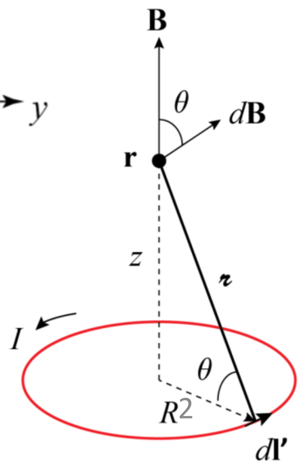
- The magnetic field due to a finite-size wire with current I

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi s}\,(\sin\theta_2 + \sin\theta_1)\hat{\mathbf{x}}$$



- The magnetic field above a loop with current I

$$\mathbf{B}(z) = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}$$



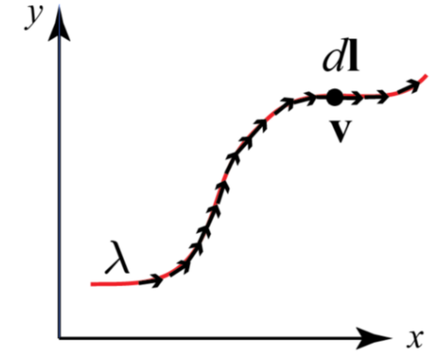
- The Ampere's Law

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} \quad\longleftrightarrow\quad \oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

Question/Clarification

- Current

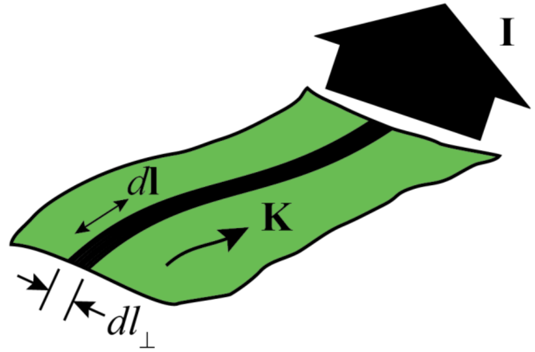
$$\mathbf{I} = \frac{\lambda\, d\mathbf{l}}{dt} = \lambda \mathbf{v}$$



- Surface Current Density

$$\mathbf{K} = \frac{d\mathbf{I}}{dl_\perp} = \sigma \mathbf{v}$$

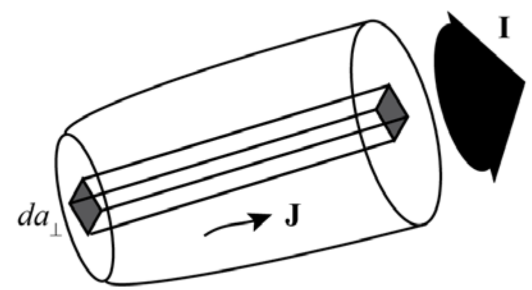
This is per unit transverse length (not area)



- Volume Current Density

$$\mathbf{J} = \frac{d\mathbf{I}}{da_\perp} = \rho \mathbf{v}$$

This is per unit transverse area (not volume)

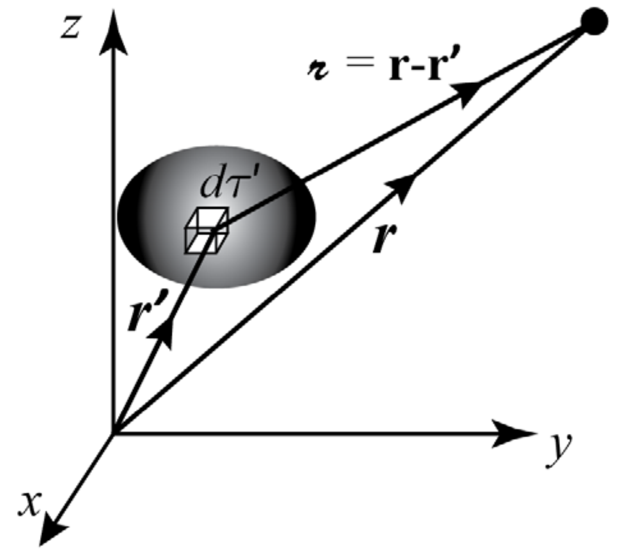


The Divergence of B

The magnetic field produced by a volume current

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} d\tau'$$

$$\boldsymbol{\mathscr{r}} = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}$$



Take the divergence of the above magnetic field (with respect to the unprimed coordinates

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \nabla \cdot \left(\frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) d\tau'$$

Using $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot \left(\nabla \times \mathbf{J}(\mathbf{r}')\right) d\tau' - \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}') \cdot \left(\nabla \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) d\tau'$$

$\nabla \times \mathbf{J}(\mathbf{r}') = \mathbf{0}$ since $\mathbf{J}(\mathbf{r}')$ does not depend on $\mathbf{r}$

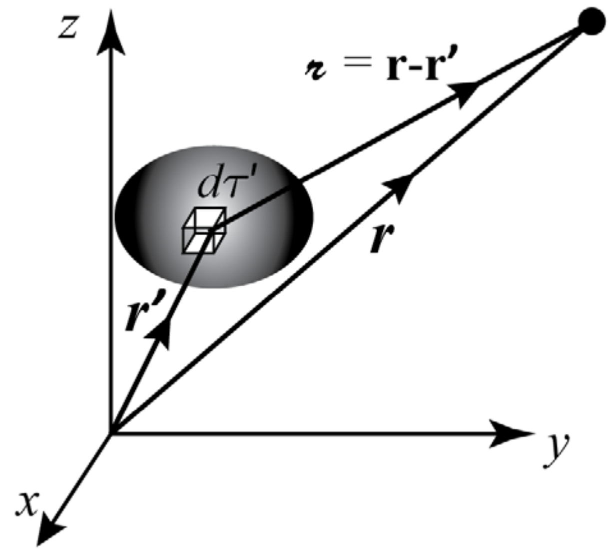
$\nabla \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = \mathbf{0}.$ (Prob. 1.62 Griffiths, 3rd ed.)

$$\boxed{\nabla \cdot \mathbf{B}(\mathbf{r}) = 0}$$

The divergence of a magnetic field is zero.

The Curl of B

The magnetic field produced by a volume current



$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

$$\mathscr{r} = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}$$

Take the curl of the above magnetic field (with respect to the unprimed coordinates

$$\nabla\times\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \nabla\times\left(\frac{\mathbf{J}(\mathbf{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$$

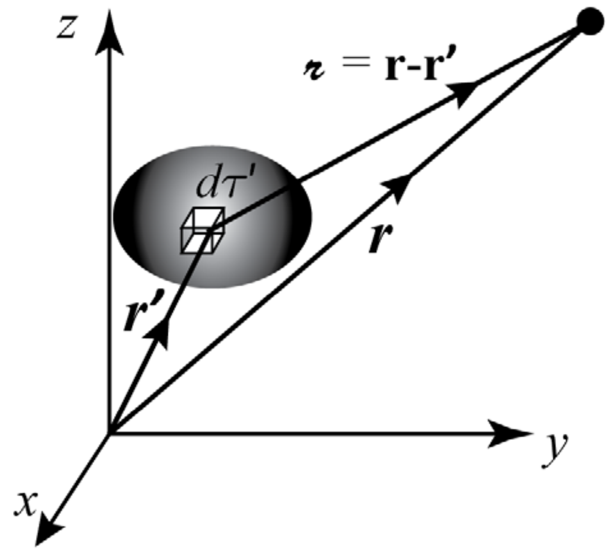
$$\nabla\times\left(\frac{\mathbf{J}(\mathbf{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\nabla\right)\mathbf{J}(\mathbf{r}') - (\mathbf{J}(\mathbf{r}')\cdot\nabla)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} + \mathbf{J}(\mathbf{r}')\left(\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) - \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\left(\nabla\cdot\mathbf{J}(\mathbf{r}')\right)$$

$\nabla\cdot\mathbf{J}(\mathbf{r}') = \mathbf{0}$ since $\mathbf{J}(\mathbf{r}')$ does not depend on $\mathbf{r}$

$\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\nabla\right)\mathbf{J}(\mathbf{r}') = \mathbf{0}$ since $\mathbf{J}(\mathbf{r}')$ does not depend on $\mathbf{r}$

$$\nabla\times\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')\left(\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau' - \frac{\mu_0}{4\pi}\int (\mathbf{J}(\mathbf{r}')\cdot\nabla)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

The Curl of B



$$\nabla \times \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')\left(\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) d\tau' - \frac{\mu_0}{4\pi}\int (\mathbf{J}(\mathbf{r}')\cdot\nabla)\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'$$

$$\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0\, \mathbf{J}(\mathbf{r}) - \frac{\mu_0}{4\pi}\int (\mathbf{J}(\mathbf{r}')\cdot\nabla)\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'$$

$$-(\mathbf{J}(\mathbf{r}')\cdot\nabla)\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = (\mathbf{J}(\mathbf{r}')\cdot\nabla')\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

$$= (\mathbf{J}(\mathbf{r}')\cdot\nabla')\left[\frac{(x-x')}{\mathscr{r}^3}\hat{\mathbf{x}} + \frac{(y-y')}{\mathscr{r}^3}\hat{\mathbf{y}} + \frac{(z-z')}{\mathscr{r}^3}\hat{\mathbf{z}}\right]$$

$$(\mathbf{J}(\mathbf{r}')\cdot\nabla')\frac{(x-x')}{\mathscr{r}^3} = \nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\right] - \frac{(x-x')}{\mathscr{r}^3}(\nabla'\cdot\mathbf{J}) = \nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\right]$$

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + (\mathbf{A}\cdot\nabla)f$$

Magnetostatics implies steady current. So $\nabla'\cdot\mathbf{J} = -\frac{d\rho}{dt} = 0$

$$\int_{Vol}\nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\right]d\tau' = \oint_{surf}\left[\frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\right]\cdot d\mathbf{a} = 0$$

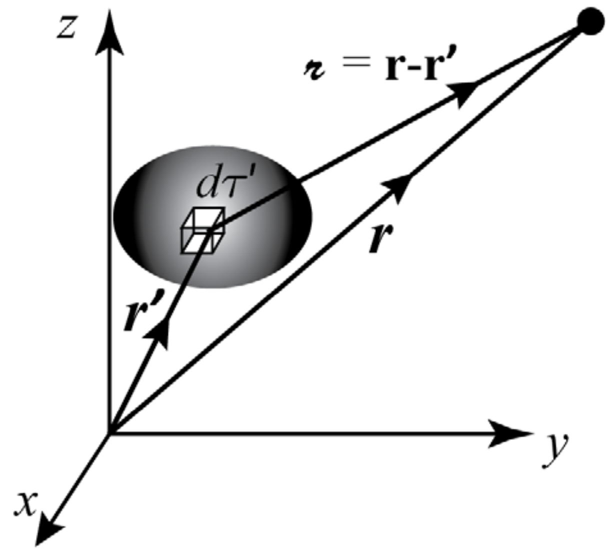
For large enough integration volume, all the currents are inside. So $\mathbf{J} = 0$ at the surface.

The Curl of B

$$\nabla \times \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')\left(\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) d\tau' - \frac{\mu_0}{4\pi}\int (\mathbf{J}(\mathbf{r}')\cdot\nabla)\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'$$

$$\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0\, \mathbf{J}(\mathbf{r}) - \frac{\mu_0}{4\pi}\int (\mathbf{J}(\mathbf{r}')\cdot\nabla)\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'$$



$$\int_{Vol} \nabla' \cdot \left[\frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\right] d\tau' = \oint_{surf}\left[\frac{(x-x')}{\mathscr{r}^3}\mathbf{J}\right]\cdot d\mathbf{a} = 0$$

For large enough integration volume, all the currents are inside. So $\mathbf{J} = 0$ at the surface.

So,

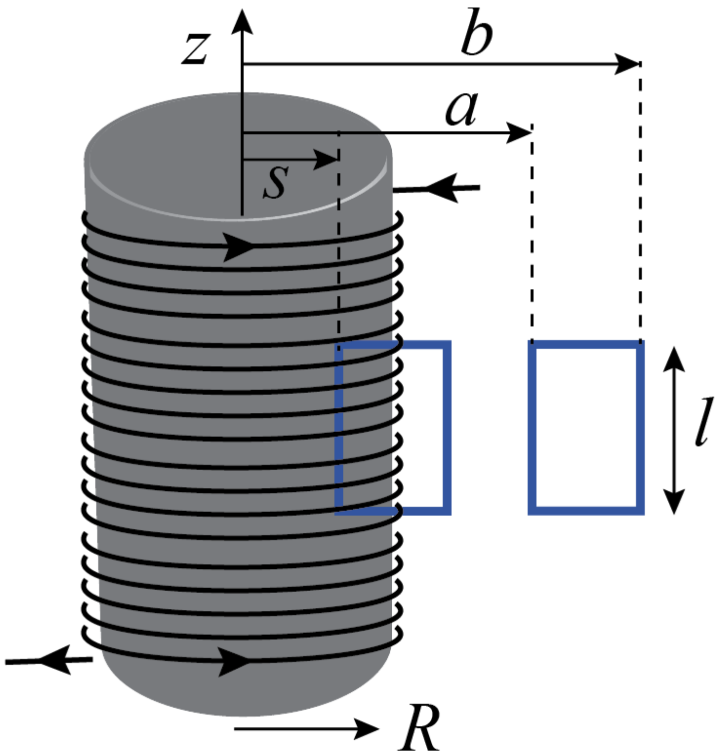
$$\frac{\mu_0}{4\pi}\int (\mathbf{J}(\mathbf{r}')\cdot\nabla)\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau' = 0$$

$$\boxed{\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0\, \mathbf{J}(\mathbf{r})}$$

Ampere's Law

Ampere's Law

Ex. 5.9 (Griffiths, 3rd Ed.): Calculate the magnetic field of a very long solenoid. (n closely wound turns per unit length, Radius R, and a steady current I.



Can there be magnetic field in the s direction? No.

Can there be magnetic field in the ϕ direction? No.

Can there be magnetic field in the z direction?

Let's check

For an Amperian loop outside

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \Rightarrow B(a)l - B(b)l = 0 \Rightarrow B(a) = \text{B}(b)$$

But since $B(\infty) = 0$, $B(a) = \text{B}(b) = \text{B}(\infty) = 0$

For an Amperian loop inside

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \Rightarrow B(s)l - 0 = \mu_0 n I l \Rightarrow B(s) = \mu_0 n I$$

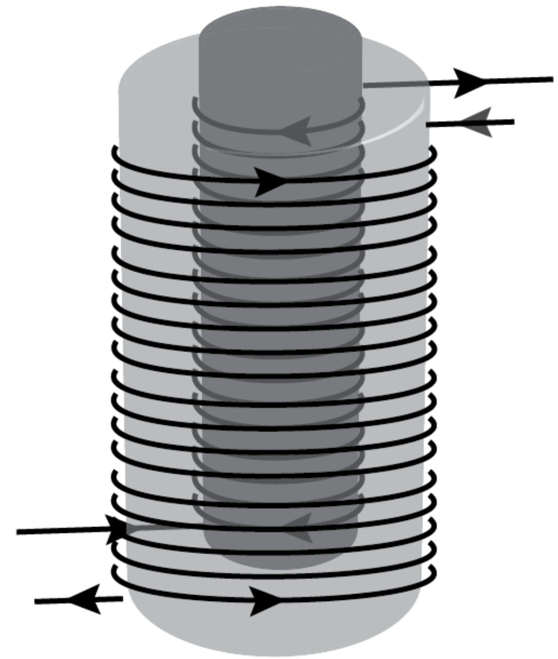
$$\mathbf{B} = 0 \quad \text{for } (s > R)$$

$$= \mu_0 n I \hat{\mathbf{z}} \quad \text{for } (s < R)$$

Magnetic field inside a solenoid is uniform---analogous to capacitor which produces uniform electric fields.

Ampere's Law

Prob 5.15 (Griffiths, 3rd Ed.): Two long coaxial solenoids. Current I flows in opposite directions. Number of turns are n_1(inner) and n_2(outer). Find magnetic field (i) inside the inner solenoid, (ii) between the solenoids, and (iii) outside both the solenoids.



Field due to a solenoid
$$\mathbf{B} = 0 \quad \text{for } (s > R)$$
$$= \mu_0 n I \hat{\mathbf{z}} \quad \text{for } (s < R)$$

Principle of linear superposition $\Rightarrow$

(i) $\mathbf{B} = \mu_0\, n_2 I \hat{\mathbf{z}} + \mu_0\, n_1 I(-\hat{\mathbf{z}}) = \mu_0\,(n_2 - n_1) I \hat{\mathbf{z}}$

(ii) $\mathbf{B} = \mu_0\, n_2 I \hat{\mathbf{z}}$

(iii) $\mathbf{B} = 0$

Magnetostatics and Electrostatics

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,\rho(\mathbf{r}')d\tau$$

Coulomb's Law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$$

Biot-Savart Law

$$\mathbf{F}_{\text{elec}} = \text{Q}\mathbf{E}$$

Electric Force

$$\mathbf{F}_{\text{mag}} = \text{Q}(\mathbf{v}\times\mathbf{B})$$

Magnetic Force

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}$$

Gauss's Law

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0$$

No Name

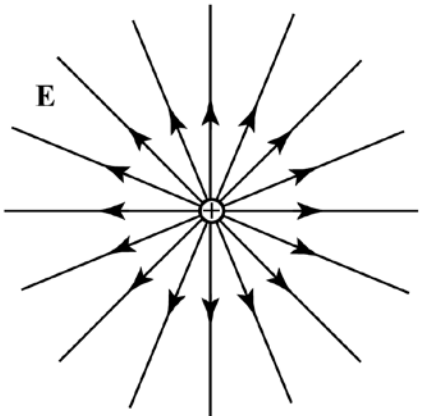
$$\boldsymbol{\nabla}\times\mathbf{E} = 0$$

No Name

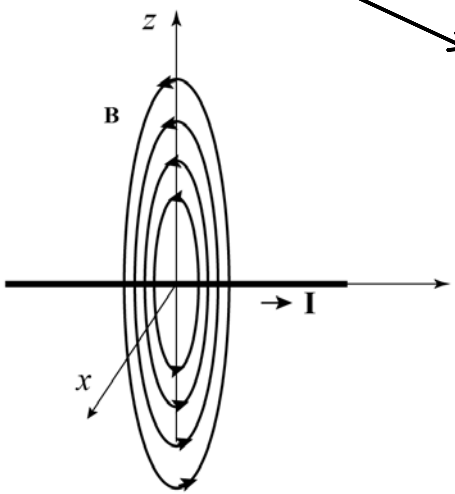
$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J}$$

Amperes's Law

Maxwell's equations (Electrostatics)



Electric field diverges away from a positive charge



Magnetic field curls around a current.

$$\boldsymbol{\nabla}\times\mathbf{E} = 0 \;\Rightarrow\; \mathbf{E} = -\boldsymbol{\nabla}\text{V}$$

Electric Potential (scalar)

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \;\Rightarrow\; \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$$

Magnetic Vector Potential

Vector Potential (From Lecture # 4):

If the divergence of a vector field $\mathbf{F}$ is zero everywhere, $(\boldsymbol{\nabla} \cdot \mathbf{F} = 0)$, then:

(1) $\int \mathbf{F} \cdot d\mathbf{a}$ is independent of surface.

(2) $\oint \mathbf{F} \cdot d\mathbf{a} = 0$ for any closed surface.

This is because of the divergence theorem

$$\int_{Vol} (\boldsymbol{\nabla} \cdot \mathbf{F}) d\tau = \oint_{Surf} \mathbf{F} \cdot d\mathbf{a}$$

(3) $\mathbf{F}$ is the curl of a vector function: $\mathbf{F} = \boldsymbol{\nabla} \times \mathbf{A}$

- This is because divergence of a curl is always zero $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = 0$
- The vector potential is not unique. A gradient $\boldsymbol{\nabla} V$ of a scalar function can be added to $\mathbf{A}$ without affecting the curl, since the curl of a gradient is zero.