Semester II, 2017-18
Department of Physics, IIT Kanpur

PHY103A: Lecture # 19

(Text Book: Intro to Electrodynamics by Griffiths, 3rd Ed.)

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Notes

- Quiz # 1 is tomorrow

Summary of Lecture # 18:

- The divergence of a magnetic field is zero.

$$\mathbf{\nabla} \cdot \mathbf{B}(\mathbf{r}) = 0$$

- The curl of a magnetic field: The Ampere's Law

$$\mathbf{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} \quad \longleftrightarrow \quad \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

- Magnetic Vector Potential

$$\mathbf{\nabla} \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = \mathbf{\nabla} \times \mathbf{A}$$

Vector Potential (From Lecture # 4):

If the divergence of a vector field $\mathbf{F}$ is zero everywhere,($\boldsymbol{\nabla} \cdot \mathbf{F} = 0$), then:

(1) $\int \mathbf{F} \cdot d\mathbf{a}$ is independent of surface.

(2) $\oint \mathbf{F} \cdot d\mathbf{a} = 0$ for any closed surface.

This is because of the divergence theorem

$$\int_{Vol} (\boldsymbol{\nabla} \cdot \mathbf{F}) d\tau = \oint_{Surf} \mathbf{F} \cdot d\mathbf{a}$$

(3) $\mathbf{F}$ is the curl of a vector function: $\mathbf{F} = \boldsymbol{\nabla} \times \mathbf{A}$

- This is because divergence of a curl is always zero $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = 0$
- The vector potential is not unique. A gradient $\boldsymbol{\nabla} V$ of a scalar function can be added to $\mathbf{A}$ without affecting the curl, since the curl of a gradient is zero.

Magnetic Vector Potential

$\nabla \cdot \mathbf{B} = 0 \;\Rightarrow\; \mathbf{B} = \nabla \times \mathbf{A}$

- $\mathbf{A}$ is the Magnetic Vector Potential
- A gradient $\nabla\lambda$ of a scalar function can be added to $\mathbf{A}$ without affecting the magnetic field.

What happens to the Ampere's Law ?

$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \;\Rightarrow\; \nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}$

$\Rightarrow\; \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$

- This is not in a very nice form.
- Ampere's law in terms of $\mathbf{B}$ seems better

- However, if we can ensure that $\nabla \cdot \mathbf{A} = 0$, we can have it in a nice form.
- This can be done since we know that a $\nabla\lambda$ can be added to $\mathbf{A}$ without changing $\mathbf{B}$

Suppose we start with $\mathbf{A}_0$, such that, $\mathbf{B} = \nabla \times \mathbf{A}_0$ but, $\nabla \cdot \mathbf{A_0} \neq \mathbf{0}$.

Then, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \;\Rightarrow\; \nabla(\nabla \cdot \mathbf{A_0}) - \nabla^2 \mathbf{A_0} = \mu_0 \mathbf{J}$

Re-define by adding $\nabla\lambda$: $\mathbf{A_0} + \nabla\lambda \equiv \mathbf{A}$ such that $\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A_0} + \nabla^2\lambda = 0$

Then $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \;\Rightarrow\; \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \;\Rightarrow\; -\nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$

Magnetic Vector Potential

What is the requirement on λ that $\nabla \cdot \mathbf{A} = 0$?? Or, $\nabla \cdot \mathbf{A_0} + \nabla^2\lambda = 0$??

For a given $\mathbf{A_0}$ the gradient λ should be such that

$$\nabla^2\lambda = -\nabla \cdot \mathbf{A_0} \quad \text{(Poisson's Equation)}$$

The solution is: $\lambda(\mathbf{r}) = \dfrac{1}{4\pi}\displaystyle\int \frac{\nabla \cdot \mathbf{A_0}}{\mathscr{r}} d\tau'$

If $\nabla \cdot \mathbf{A_0} \to \mathbf{0}$, when $\mathbf{r} \to \infty$.

Recall: $\nabla^2 V = -\dfrac{\rho}{\epsilon_0}$ (Poisson's Equation)

The solution is: $V(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau'$

If the localized charge distribution $\rho(\mathbf{r}') \to \mathbf{0}$, when $\mathbf{r} \to \infty$.

Thus, one can always redefine the vector potential such that $\nabla \cdot \mathbf{A} = 0$

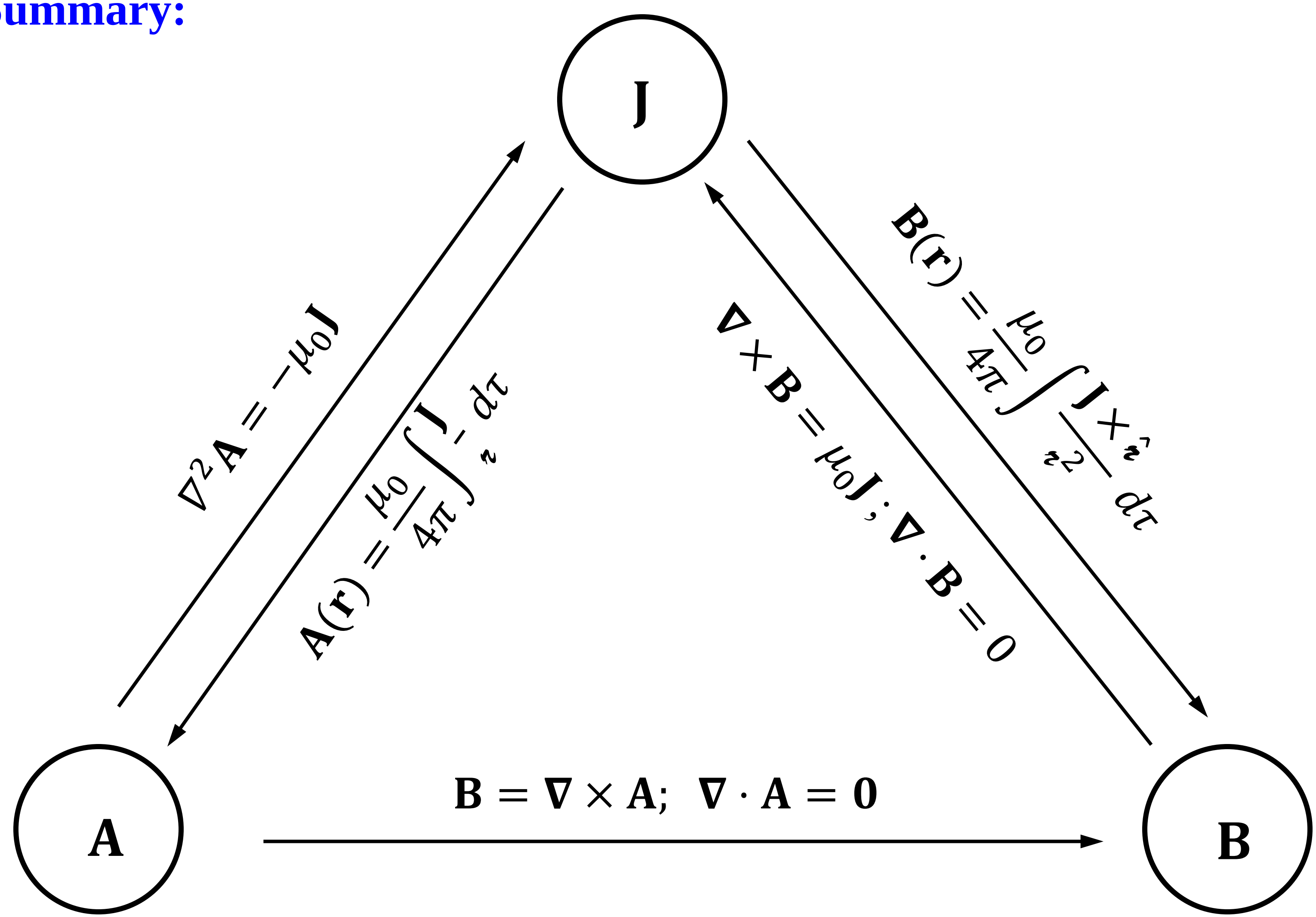
So, the Ampere's law can be written as $\boxed{-\nabla^2\mathbf{A} = \mu_0\mathbf{J}}$ It is three Poisson's Equations

So, $\mathbf{A}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} d\tau'$ This is simpler than Biot-Savart Law.

For surface current: $\mathbf{A}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{\mathbf{K}(\mathbf{r}')}{\mathscr{r}} da'$

For line current: $\mathbf{A}(\mathbf{r}) = \dfrac{\mu_0}{4\pi} I \displaystyle\int \frac{d\mathbf{l}'}{\mathscr{r}}$

Summary:



Magnetostatics

Magnetostatic Boundary Conditions (Consequences of the fundamental laws):

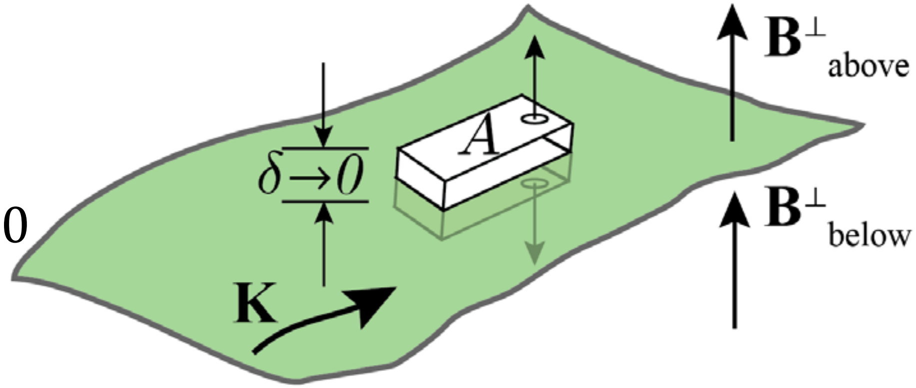
How does magnetic field (**B**) change across a boundary containing surface current **K**?

1. Normal component of B is continuous

$$\nabla \cdot \mathbf{B} = 0 \longleftrightarrow \oint_{surf} \mathbf{B} \cdot d\mathbf{a} = 0$$

$$\mathbf{B}^{\perp}{}_{above} A - \mathbf{B}^{\perp}{}_{below} A + 0 + 0 + 0 + 0 = 0$$

$$\boxed{B^{\perp}{}_{above} = B^{\perp}{}_{below}}$$

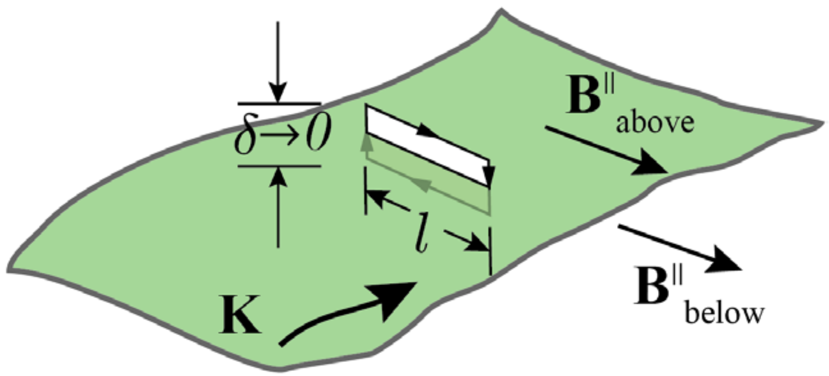


2. Parallel component of B is Discontinuous

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \longleftrightarrow \oint_{path} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \mathbf{I}_{enc}$$

$$\mathbf{B}^{||}{}_{above}\, l - \mathbf{B}^{||}{}_{below}\, l + 0 + 0 = \mu_0 K l$$

$$\boxed{B^{||}{}_{above} - B^{||}{}_{below} = \mu_0 K}$$



$$(B^{\perp}{}_{above} - B^{\perp}{}_{below})\hat{\mathbf{n}} + (B^{||}{}_{above} - B^{||}{}_{below})\widehat{\mathbf{n}^{||}} = \mu_0 \mathbf{K} \times \hat{\mathbf{n}}$$

$$\boxed{\mathbf{B}_{above} - \mathbf{B}_{below} = \mu_0 \mathbf{K} \times \hat{\mathbf{n}}}$$

Magnetostatic Boundary Conditions (Consequences of the fundamental laws):

How does the magnetic potential (**A**) change across a boundary containing surface current **K**?

1. Normal component of **A** is continuous

$$\nabla \cdot \mathbf{A} = 0 \longleftrightarrow \oint_{surf} \mathbf{A} \cdot d\mathbf{a} = 0$$

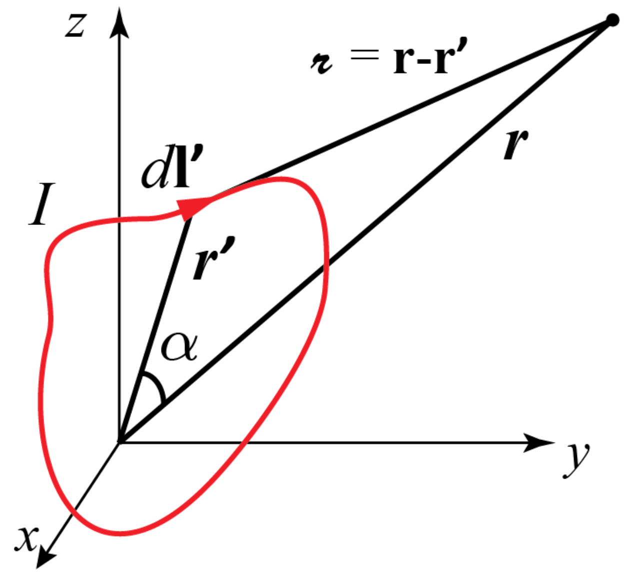
$$\boxed{A^{\perp}{}_{\text{above}} = A^{\perp}{}_{\text{below}}}$$

2. Parallel component of **A** is continuous

$$\nabla \times \mathbf{A} = \mathbf{B} \longleftrightarrow \oint_{path} \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{a} = 0$$

$$\boxed{A^{||}{}_{\text{above}} = A^{||}{}_{\text{below}}}$$

Multipole Expansion of the Vector Potential



Using the cosine rule,

$$\mathscr{r}^2 = r^2 + r'^2 - 2rr'\cos\alpha$$

$$\mathscr{r}^2 = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

Source coordinates: (r', θ', ϕ')

Observation point coordinates: (r, θ, ϕ)

Angle between **r** and **r**$'$: α

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{d\mathbf{l}'}{\mathscr{r}}$$

$$= \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n \; P_n(\cos\alpha) d\mathbf{l}'$$

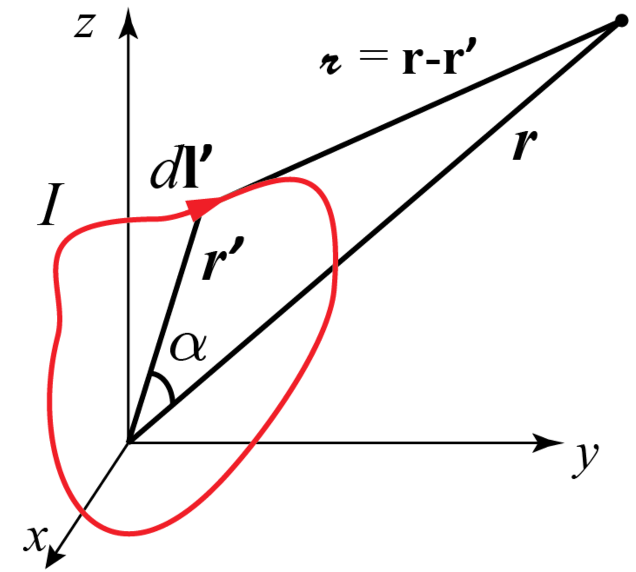
$$= \frac{\mu_0 I}{4\pi}\frac{1}{r}\oint d\mathbf{l}' + \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint r'\cos\alpha \, d\mathbf{l}' + \frac{\mu_0 I}{4\pi}\frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos\alpha - \frac{1}{2}\right) d\mathbf{l}' + \cdots$$

Monopole potential ($1/r$ dependence)

Dipole potential ($1/r^2$ dependence)

Quadrupole potential ($1/r^3$ dependence)

Multipole Expansion of the Vector Potential



Monopole potential

$$\mathbf{A}_{mono}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r}\oint d\mathbf{l}' \quad = 0$$

Dipole potential

$$\mathbf{A}_{dip}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint r'\cos\alpha\, d\mathbf{l}' \quad = \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint (\hat{\mathbf{r}}\cdot\mathbf{r}')d\mathbf{l}'$$

Source coordinates: (r', θ', ϕ')

Observation point coordinates: (r, θ, ϕ)

Angle between **r** and **r**′: α

Stokes Theorem: $$\int_{Surf}(\nabla\times\mathbf{V})\cdot d\mathbf{a} = \oint_{Path}\mathbf{V}\cdot d\mathbf{l}$$

Using Stokes's theorem for a vector **V** that can be written as $\mathbf{V} = \mathbf{c}T$, where **c** is a constant vector

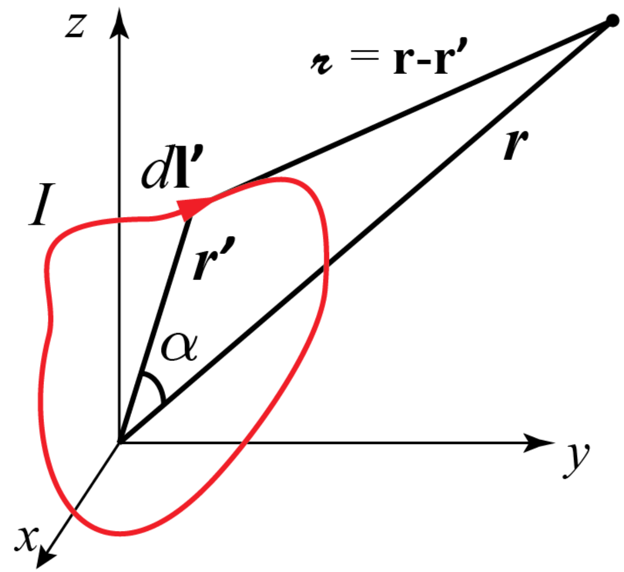
$$\int_{Surf}(\nabla\times\mathbf{c}T)\cdot d\mathbf{a} = \oint_{Path}\mathbf{c}T\cdot d\mathbf{l}$$

$$\int_{Surf}T(\nabla\times\mathbf{c})\cdot d\mathbf{a} - \int_{Surf}(\mathbf{c}\times\nabla T)\cdot d\mathbf{a} \quad = \mathbf{c}\cdot\oint_{Path}T d\mathbf{l}$$

Or, $$-\int_{Surf}(\mathbf{c}\times\nabla T)\cdot d\mathbf{a} = \mathbf{c}\cdot\oint_{Path}T d\mathbf{l}$$ Or, $$-\mathbf{c}\cdot\int_{Surf}\nabla T\times d\mathbf{a} = \mathbf{c}\cdot\oint_{Path}T d\mathbf{l}$$

Therefore, $$\oint_{path}T d\mathbf{l} \; = -\int_{Surf}\nabla T\times d\mathbf{a}$$

Multipole Expansion of the Vector Potential



Monopole potential

$$\mathbf{A}_{\text{mono}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r}\oint d\mathbf{l}' \quad = 0$$

Dipole potential

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint r'\cos\alpha\, d\mathbf{l}' \; = \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint (\hat{\mathbf{r}}\cdot\mathbf{r}')d\mathbf{l}'$$

[Corollary of Stokes Theorem: $\oint_{path} T d\mathbf{l} = -\int_{Surf} \nabla T \times d\mathbf{a}$]

$$= -\frac{\mu_0 I}{4\pi}\frac{1}{r^2}\int \nabla'(\hat{\mathbf{r}}\cdot\mathbf{r}') \times d\mathbf{a}'$$

Source coordinates: (r', θ', ϕ')

Observation point coordinates: (r, θ, ϕ)

Angle between $\mathbf{r}$ and $\mathbf{r}'$: α

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = -\frac{\mu_0 I}{4\pi}\frac{1}{r^2}\int \hat{\mathbf{r}} \times d\mathbf{a}' \; = \frac{\mu_0}{4\pi}\frac{1}{r^2}\left(I\int d\mathbf{a}'\right)\times\hat{\mathbf{r}} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$$

$$\mathbf{m} \equiv I\int d\mathbf{a}'$$

Magnetic dipole moment

$\int d\mathbf{a}'$
- is the vector area of the loop
- is the scalar area if the loop is flat

Magnetic field due to a magnetic dipole

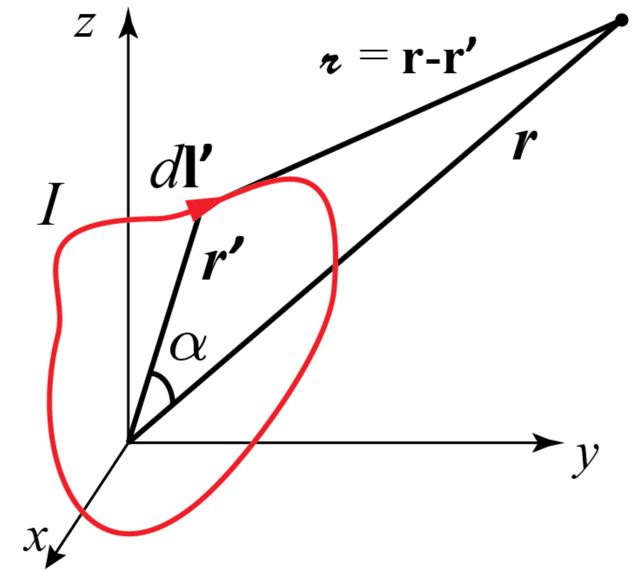
$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$$

Take $\mathbf{m} = m\,\hat{\mathbf{z}}$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}}$$

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \boldsymbol{\nabla}\times\mathbf{A}_{\text{dip}}(\mathbf{r})$$

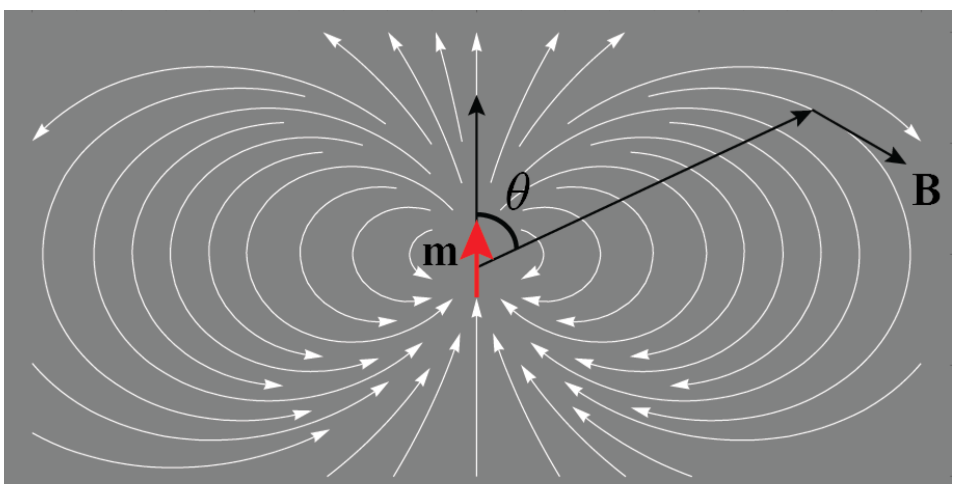
$$= \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$$



Source coordinates: (r', θ', ϕ')

Observation point coordinates: (r, θ, ϕ)

Angle between $\mathbf{r}$ and $\mathbf{r}'$: α



Recall

$\mathbf{p} = p\hat{\mathbf{z}}$

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta}\right)$$