Semester II, 2017-18
Department of Physics, IIT Kanpur

PHY103A: Lecture # 2

(Text Book: Introduction to Electrodynamics by David J Griffiths)

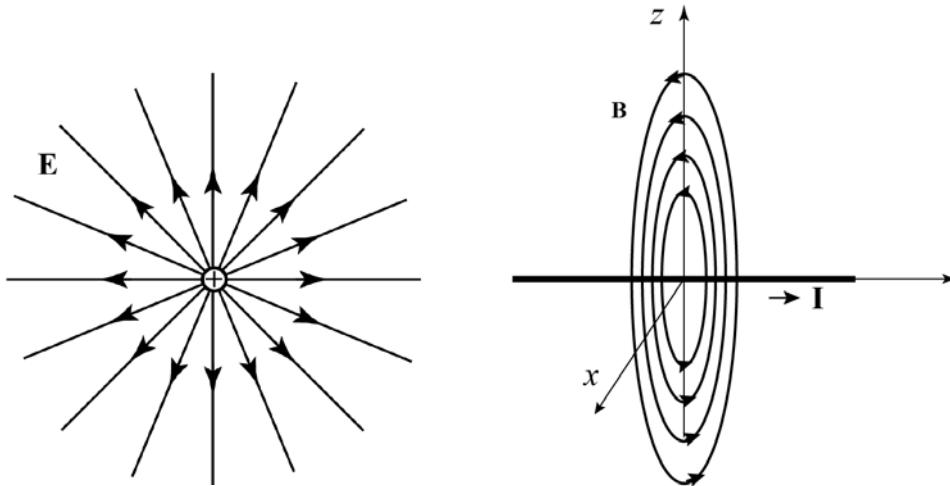Anand Kumar Jha
06-Jan-2018

Notes

- Homework # 1 is uploaded on the course webpage

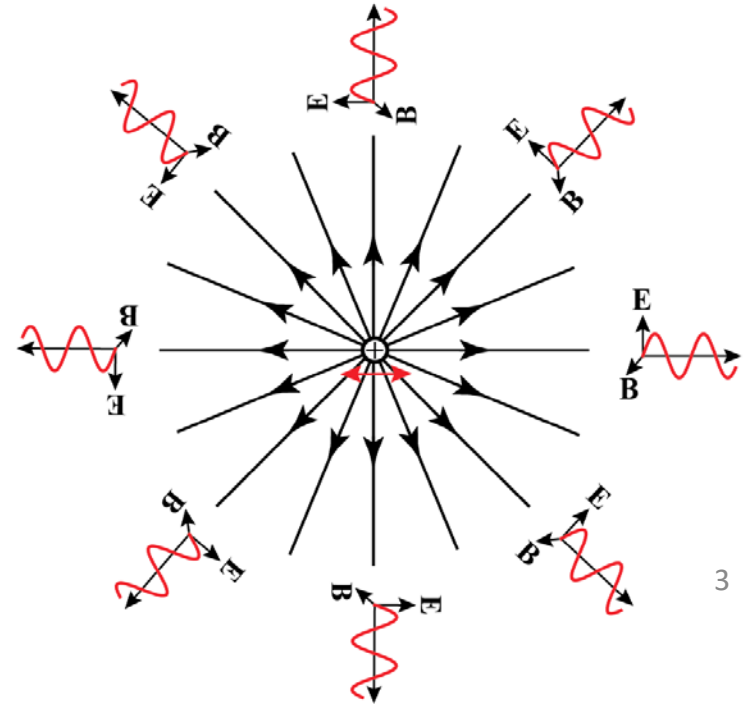- Lecture # 1 has also been uploaded on the course webpage

- Course Webpage: http://home.iitk.ac.in/~akjha/PHY103.htm

Summary of Lecture # 1:
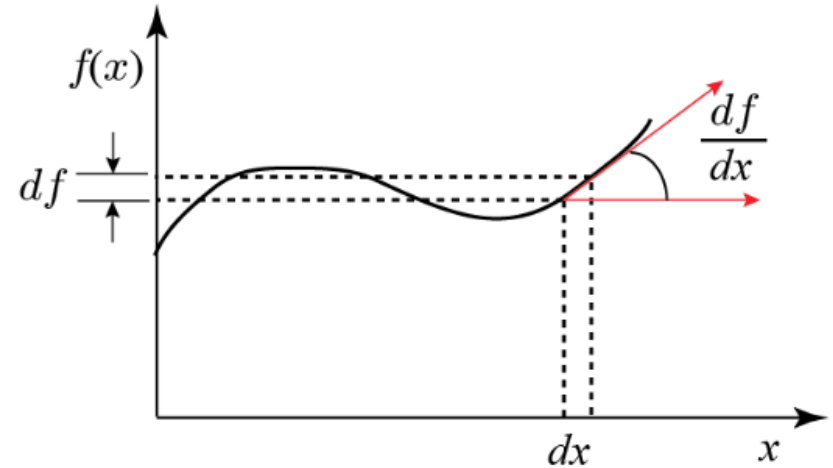
- **Electrostatics**

Electrodynamics

- **Scalars and Vectors**

- **Scalar and vector functions/fields**

- **Vector Algebra**

- **Vector calculus**

Differential Calculus (of function $f(x)$ of single variable x.)

$$df = \left(\frac{df}{dx}\right) dx$$

Change in $f(x)$

Derivative

Change in x

- The magnitude of $\dfrac{df}{dx}$ is the rate of change (slope) of function $f(x)$

- If $\dfrac{df}{dx} = 0$ then $df = 0$, this defines the extremum of function $f(x)$.

Vector Calculus

Differential Calculus (of function $T(x, y, z)$ of three variables.)

Q: How do we find out the rate of change of a function of more than one variable?
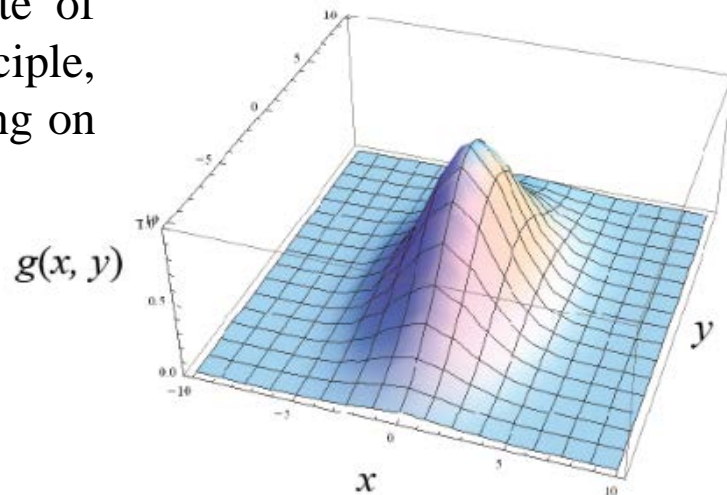
Issue: For a function of more than one variable the rate of change depends on the direction we move. So, in principle, there can be infinite number of "rate of changes" depending on the direction.

Solution:

It is known from a theorem on partial derivatives that a function of three variables can be written as

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

$g(x, y)$

- This shows how T changes when the three variables are changed by the infinitesimal amounts dx, dy, dz.

- The above representation means that one does not require an infinite number of "slopes" or "rate of changes." It is sufficient to know just three.

Differential Calculus (of function $T(x, y, z)$ of three variables.)

Q: How do we find out the rate of change of a function of more than one variable?

Solution: We can rewrite dT as:

$$dT = \left(\frac{\partial T}{\partial x}\widehat{\boldsymbol{x}} + \frac{\partial T}{\partial y}\widehat{\boldsymbol{y}} + \frac{\partial T}{\partial z}\widehat{\boldsymbol{z}}\right) \cdot (dx\,\widehat{\boldsymbol{x}} + dy\,\widehat{\boldsymbol{y}} + dz\,\widehat{\boldsymbol{z}})$$

$$dT = (\boldsymbol{\nabla}T) \cdot (d\mathbf{l})$$

Change in T

3-dimensional derivative

displacement vector

$$\boxed{df = \left(\frac{df}{dx}\right)dx}$$

Derivative

- $\boldsymbol{\nabla}T \equiv \frac{\partial T}{\partial x}\widehat{\boldsymbol{x}} + \frac{\partial T}{\partial y}\widehat{\boldsymbol{y}} + \frac{\partial T}{\partial z}\widehat{\boldsymbol{z}}$ is the generalized derivative

- $\boldsymbol{\nabla}T$ is called the gradient of T. It is a vector quantity with three components.

- $dT = (\boldsymbol{\nabla}T) \cdot (d\mathbf{l}) = |\boldsymbol{\nabla}T||d\mathbf{l}| \cos\theta$. For a fixed $|d\mathbf{l}|$, dT is maximum when $\theta = 0$

- So, $\boldsymbol{\nabla}T$ points in the direction of maximum increase of the function T.

- The magnitude $|\boldsymbol{\nabla}T|$ is the slope or the rate of change along this maximal direction.

- If $\boldsymbol{\nabla}T = 0$, $dT = 0$. So, $\boldsymbol{\nabla}T = 0$ defines the extremum of the function T.

Differential Calculus (of function $T(x, y, z)$ of three variables.)

Example # 1

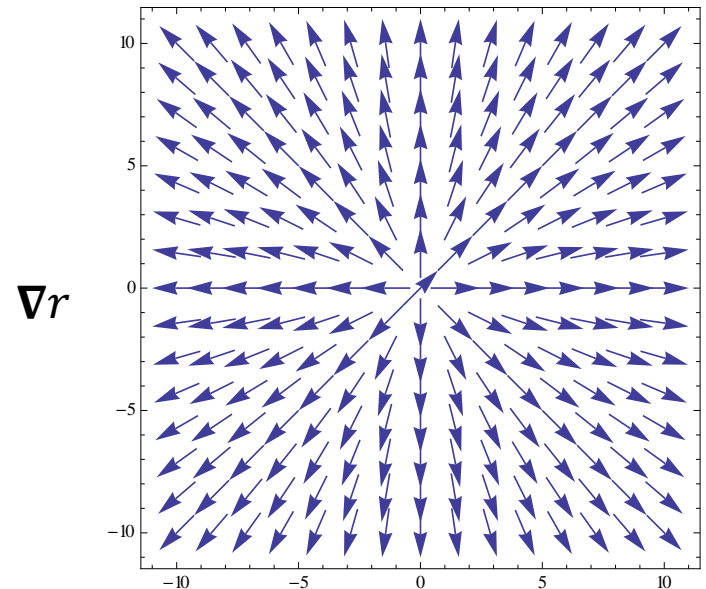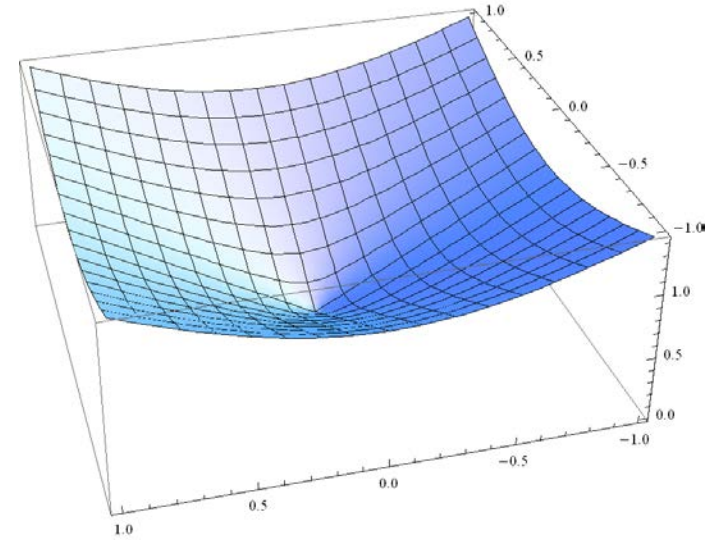- Find the gradient of $r(x, y) = \sqrt{x^2 + y^2}$

$$\nabla r = \frac{\partial r}{\partial x}\widehat{x} + \frac{\partial r}{\partial y}\widehat{y}$$

$$= \frac{1}{2}\frac{2x}{\sqrt{x^2 + y^2}}\widehat{x} + \frac{1}{2}\frac{y}{\sqrt{x^2 + y^2}}\widehat{y}$$

$$= \frac{x}{r}\widehat{x} + \frac{y}{r}\widehat{y}$$
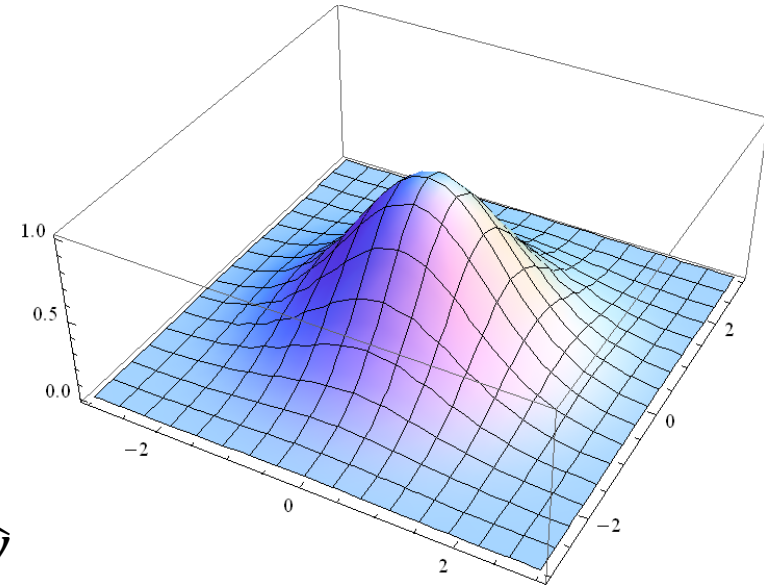
$$= \widehat{r}$$

∇r

Differential Calculus (of function $T(x, y, z)$ of three variables.)

Example # 2

- Find the gradient of $g(x, y) = \exp\left[-\dfrac{x^2 + y^2}{2}\right]$

$$\boldsymbol{\nabla} g = \frac{\partial g}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial g}{\partial y}\hat{\boldsymbol{y}}$$

$$= -x \exp\left[-\frac{x^2 + y^2}{2}\right]\hat{\boldsymbol{x}} - y \exp\left[-\frac{x^2 + y^2}{2}\right]\hat{\boldsymbol{y}}$$
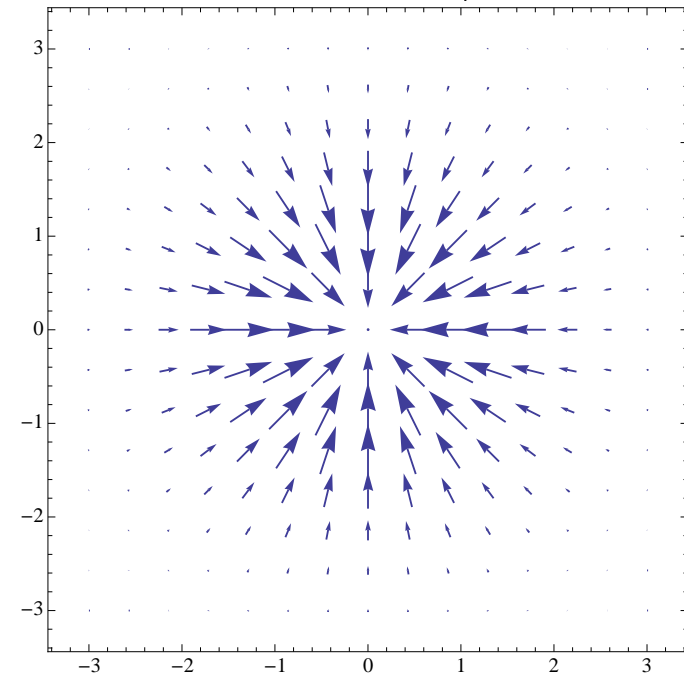

$$\boldsymbol{\nabla} g(x, y)$$

- Find the extremum of $g(x, y)$

$$\boldsymbol{\nabla} g = 0 \qquad \text{One solution is } (x, y) = (0, 0)$$

So, the function has an extremum at $(0, 0)$, which is a maximum.

The gradient operator

$$\nabla T \equiv \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)T$$

So, $\quad \nabla \equiv \dfrac{\partial}{\partial x}\hat{x} + \dfrac{\partial}{\partial y}\hat{y} + \dfrac{\partial}{\partial z}\hat{z}$

- ∇ is called the gradient operator.

- ∇ is not a vector in the usual sense, but a vector operator.

- ∇ acts on a scalar function and gives out the generalized derivative

If we are dealing with scalar functions then the differential calculus consists of derivatives only. However, in the case of vector functions/fields, the differential calculus has two more concepts, namely, the two vector derivatives.

Divergence of a vector $\quad \nabla \cdot \mathbf{V}$

Curl of a vector $\qquad \nabla \times \mathbf{V}$

The Divergence

The divergence of a vector **V** is defined as

$$\boldsymbol{\nabla} \cdot \mathbf{V} = \left(\frac{\partial}{\partial x}\widehat{\boldsymbol{x}} + \frac{\partial}{\partial y}\widehat{\boldsymbol{y}} + \frac{\partial}{\partial z}\widehat{\boldsymbol{z}} \right) \cdot (v_x\widehat{\boldsymbol{x}} + v_y\widehat{\boldsymbol{y}} + v_z\boldsymbol{z})$$

$$= \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)$$

- The divergence of a vector is a scalar quantity.

- The divergence measures how much a vector field diverges.

The Divergence
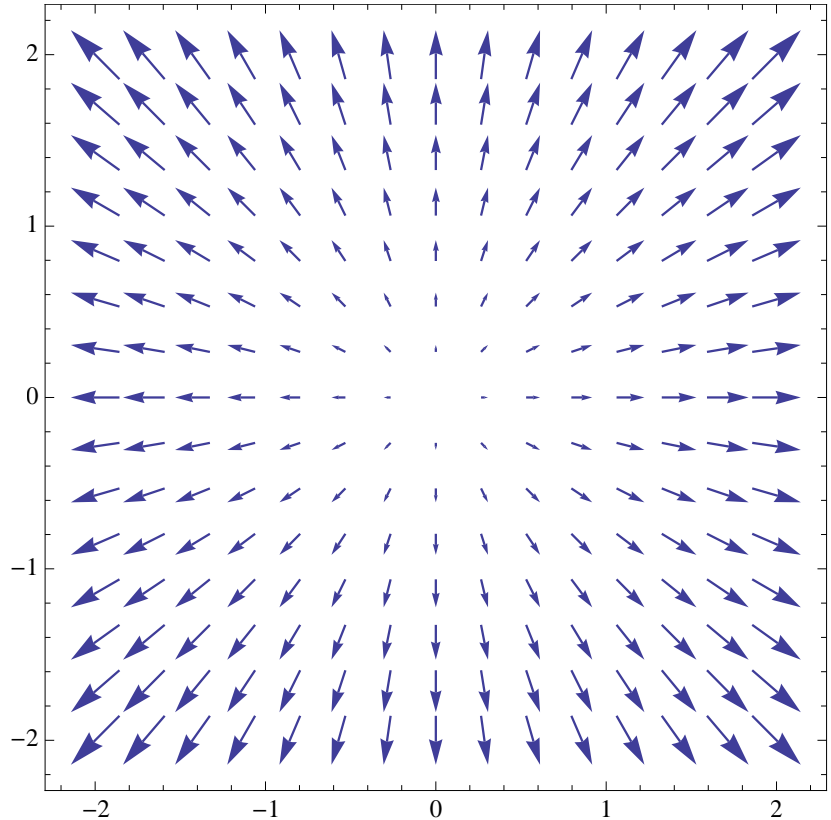
Example # 1

- Find the divergence of $\mathbf{V} = x\,\hat{x} + y\,\hat{y}$

$$\boldsymbol{\nabla} \cdot \mathbf{V} = \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$$
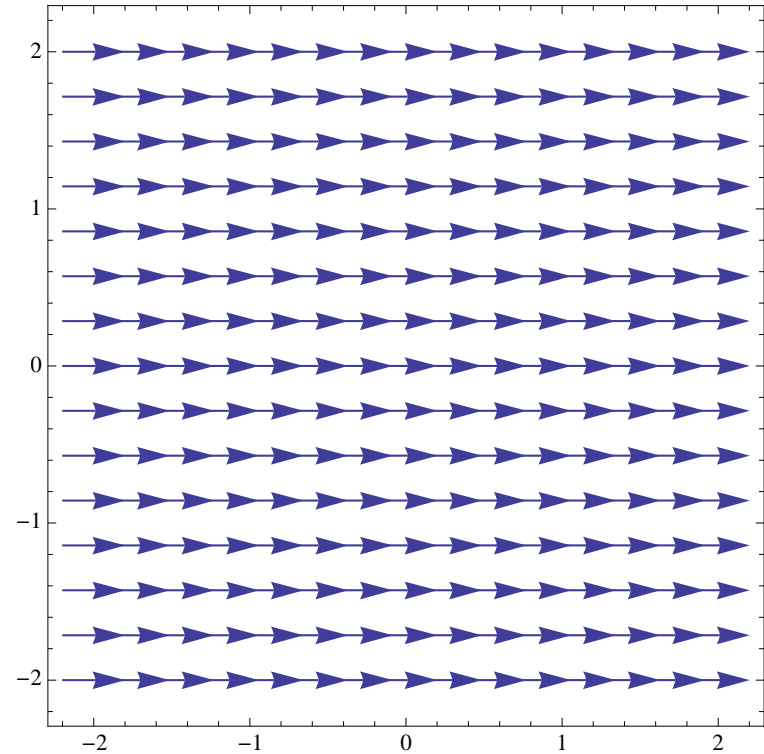
$$= 1 + 1 = 2$$

The Divergence

Example # 2

- Find the divergence of $\mathbf{V} = \hat{x}$

$$\boldsymbol{\nabla} \cdot \mathbf{V} = \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)$$

$$= 0$$

The Divergence

Example # 3

- Find the divergence of $\mathbf{V} = x\,\hat{x}$

$$\boldsymbol{\nabla} \cdot \mathbf{V} = \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)$$
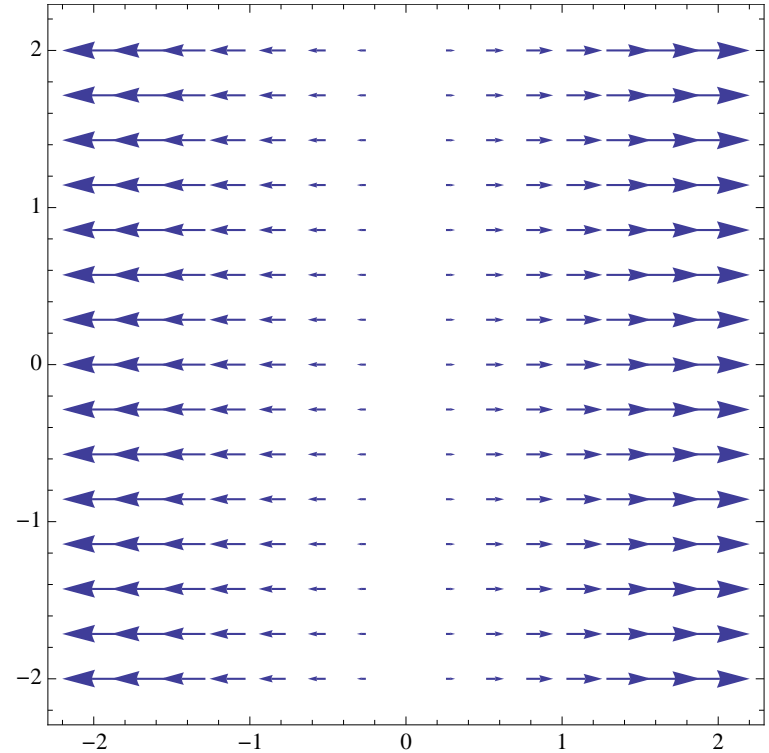
$$= 1$$

The Divergence

Example # 1

- Find the divergence of $\mathbf{V} = -y\,\hat{x} + x\,\hat{y}$

$$\boldsymbol{\nabla} \cdot \mathbf{V} = \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)$$
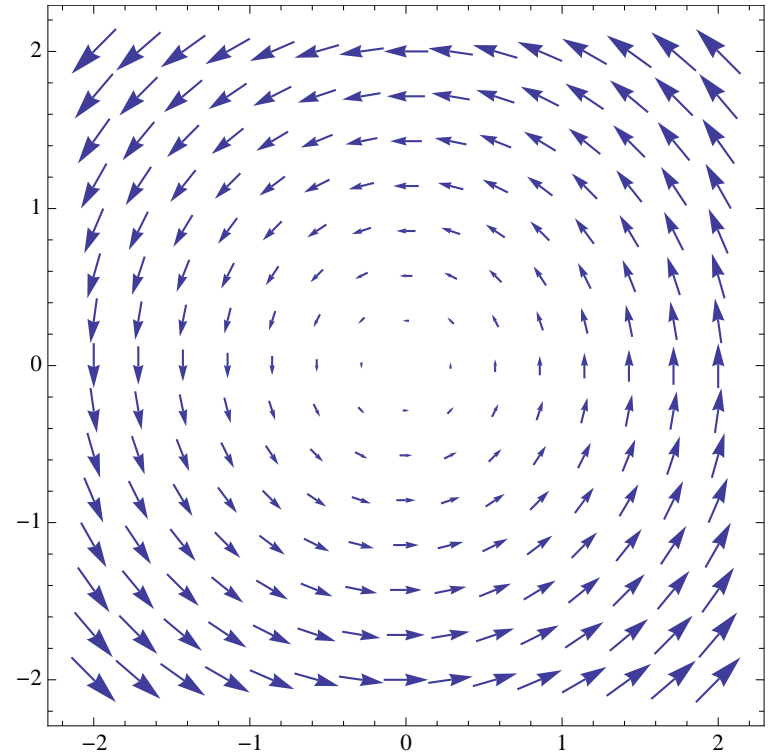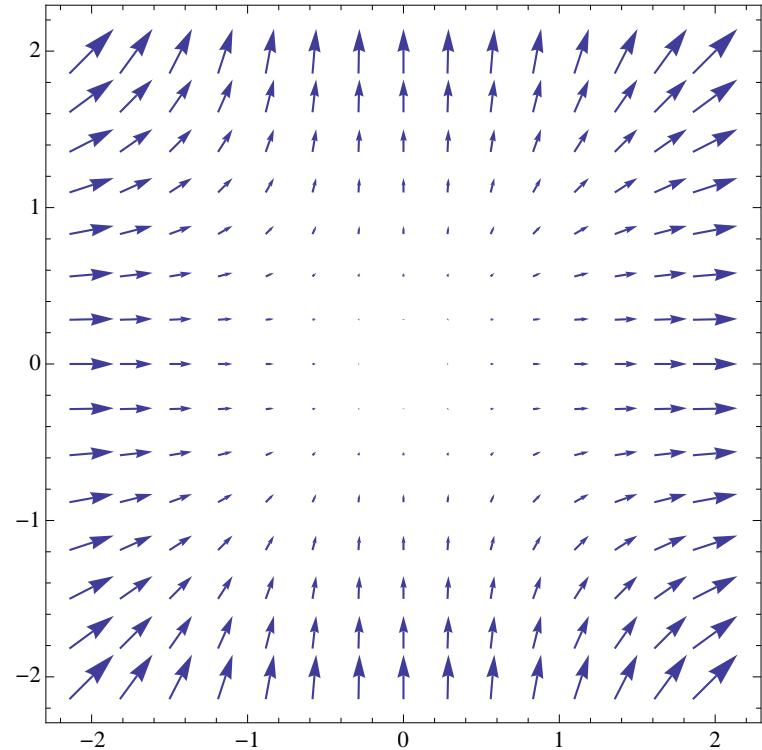
$$= 0 + 0 = 0$$

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The Divergence

Example # 5

- Find the divergence of $\mathbf{V} = x^2\,\hat{x} + y^2\,\hat{y}$

$$\mathbf{\nabla} \cdot \mathbf{V} = \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)$$

$$= 2\,x + 2\,y$$

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The Curl

The curl of a vector **V** is defined as

$$\nabla \times \mathbf{V} = \left(\frac{\partial}{\partial x} \widehat{\boldsymbol{x}} + \frac{\partial}{\partial y} \widehat{\boldsymbol{y}} + \frac{\partial}{\partial z} \widehat{\boldsymbol{z}} \right) \times (v_x \widehat{\boldsymbol{x}} + v_y \widehat{\boldsymbol{y}} + v_z \boldsymbol{z})$$

$$= \begin{vmatrix} \widehat{\boldsymbol{x}} & \widehat{\boldsymbol{y}} & \widehat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \widehat{\boldsymbol{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \widehat{\boldsymbol{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \widehat{\boldsymbol{z}}$$

- The curl of a vector is a vector quantity.
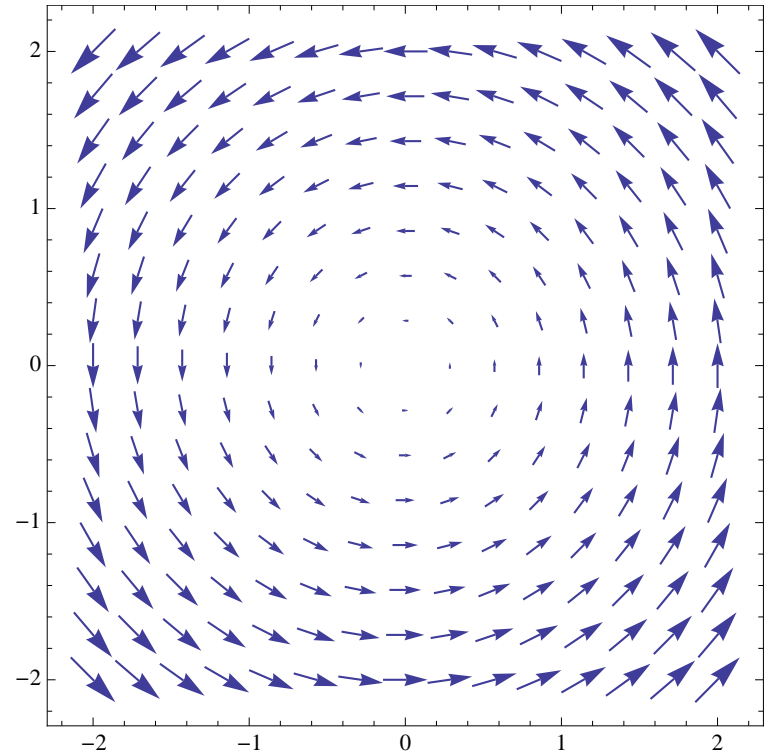
- The curl measures how much a vector field curls.

The Curl

Example # 1

- Find the curl of $\mathbf{V} = -y\,\hat{x} + x\,\hat{y}$

$$\boldsymbol{\nabla} \times \mathbf{V} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z}$$

$$= 2\hat{z}$$

The Curl

Example # 2

- Find the curl of $\mathbf{V} = x\,\hat{x} + y\,\hat{y}$

$$\mathbf{\nabla} \times \mathbf{V} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z}$$

$$= 0$$

The Curl
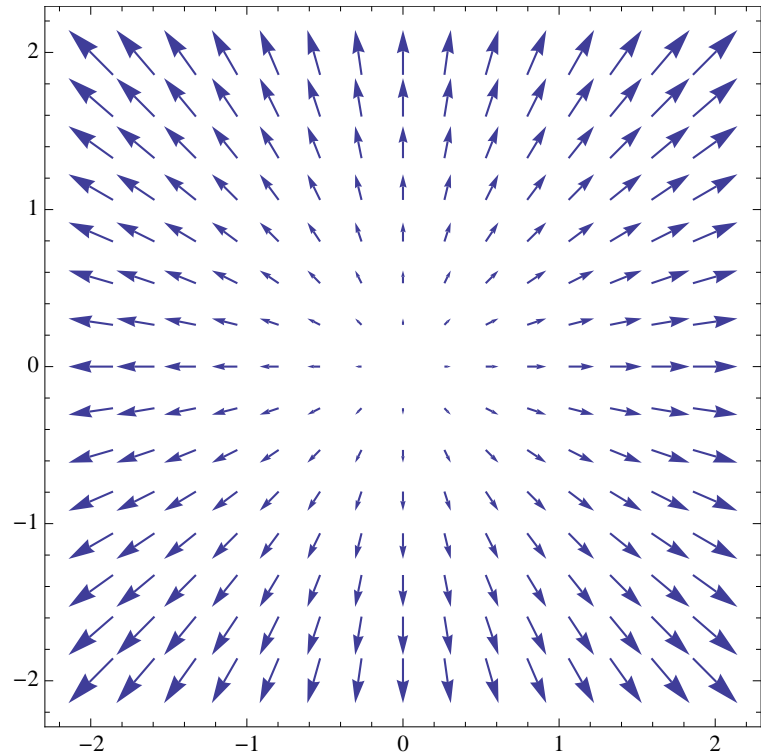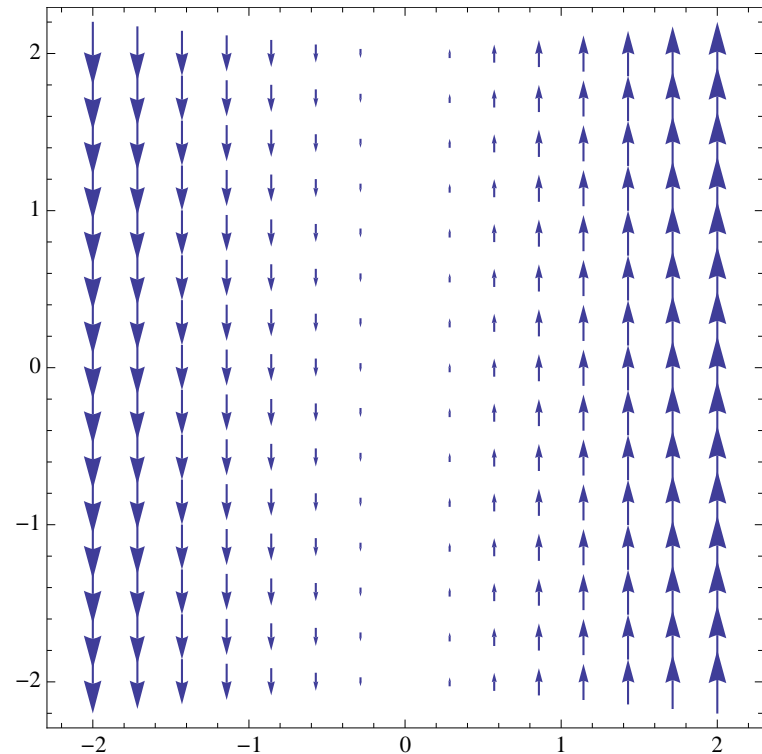
Example # 3

- Find the curl of $\mathbf{V} = x\,\hat{y}$

$$\boldsymbol{\nabla} \times \mathbf{V} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\boldsymbol{z}}$$
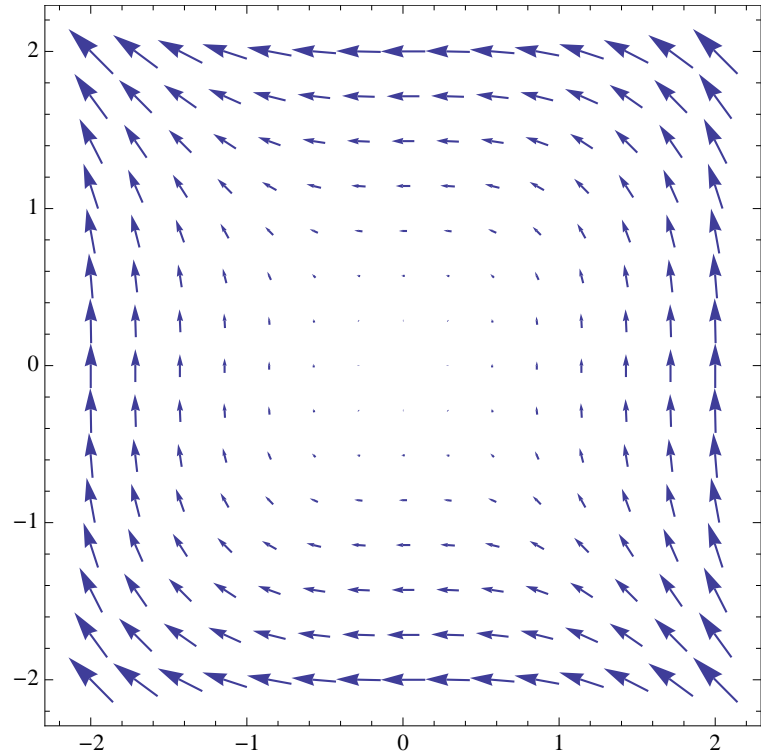
$$= \hat{\boldsymbol{z}}$$

The Curl

Example # 4

- Find the curl of $\mathbf{V} = -y^2\,\hat{x} + x^2\,\hat{y}$

$$\boldsymbol{\nabla} \times \mathbf{V} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\boldsymbol{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\boldsymbol{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\boldsymbol{z}}$$

$$= 2(x + y)\,\hat{\boldsymbol{z}}$$

Vector calculus with gradient operator

Products Rules:

1. $\nabla(fg) = f(\nabla g) + g(\nabla f)$

2. $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

3. $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

4. $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

5. $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

6. $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives:

1. $\nabla \cdot (\nabla f) = \nabla^2 f$

2. $\nabla \times (\nabla f) = 0$

3. $\nabla(\nabla \cdot \mathbf{A})$

4. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

5. $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

- $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

- ∇^2 is called the Laplacian