Semester II, 2017-18
Department of Physics, IIT Kanpur

PHY103A: Lecture # 20

(Text Book: Intro to Electrodynamics by Griffiths, 3rd Ed.)

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Notes

- Mid-sem: Feb 22, Thursday, 8-10 am
- Course coverage up to class on Monday (Feb 12)
- Q1: 10; Q2=10; Midsem: 80; Endsem: 120
- Regraded Quiz # 1 will be returned after midsem

Summary of Lecture # 19:

- **Magnetic Vector Potential**

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$

One can always redefine the vector potential such that its divergence is zero $\nabla \cdot \mathbf{A} = 0$

$$\Rightarrow \quad -\nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \quad \Rightarrow \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\, d\tau'$$

- **Magnetostatic Boundary Conditions (Consequences of the fundamental laws):**

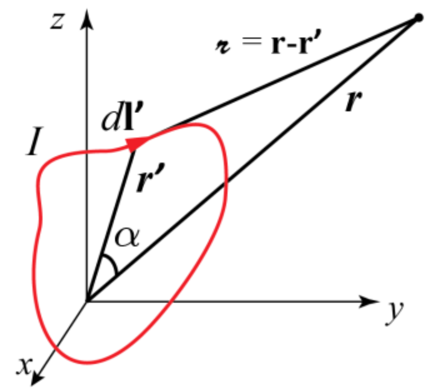
$$B^{\perp}{}_{\text{above}} = B^{\perp}{}_{\text{below}} \qquad B^{||}{}_{\text{above}} - B^{||}{}_{\text{below}} = \mu_0 K$$

$$A^{\perp}{}_{\text{above}} = A^{\perp}{}_{\text{below}} \qquad A^{||}{}_{\text{above}} = A^{||}{}_{\text{below}}$$

- **Multipole Expansion of vector potential**

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r}\oint d\mathbf{l}' + \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint r' \cos\alpha \, d\mathbf{l}'$$

$$+ \frac{\mu_0 I}{4\pi}\frac{1}{r^3}\oint (r')^2 \left(\frac{3}{2}\cos\alpha - \frac{1}{2}\right) d\mathbf{l}' + \cdots$$



- **Magnetic dipole moment**

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} \qquad \mathbf{m} \equiv I\int d\mathbf{a}'$$

Magnetic Vector Potential:

Prob. 5.23 (Griffiths, 3rd Ed.): What is the current density $\mathbf{J}$ that would produce the magnetic potential $\mathbf{A} = k\hat{\boldsymbol{\phi}}$

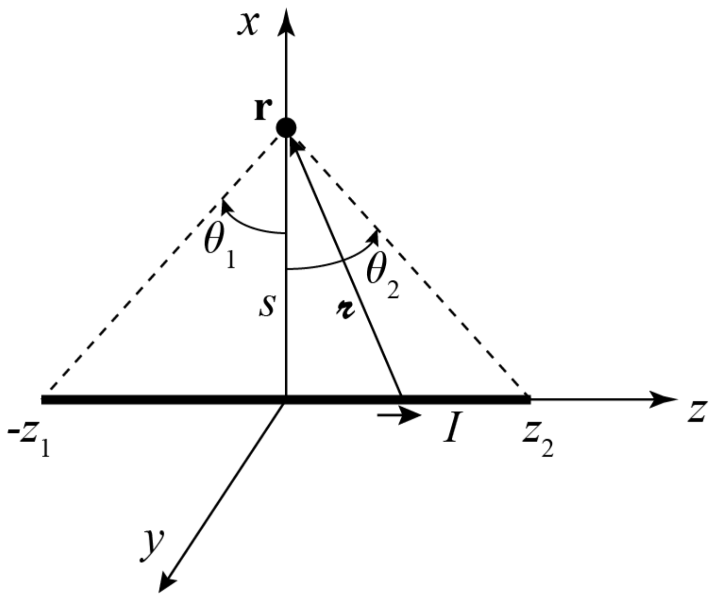
$$\mathbf{A} = k\hat{\boldsymbol{\phi}}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{s}\frac{\partial}{\partial s}(sk)\hat{\mathbf{z}} = \frac{k}{s}\,\hat{\mathbf{z}}$$

$$\mathbf{J} = \frac{1}{\mu_0}\,(\boldsymbol{\nabla} \times \mathbf{B}) = \frac{1}{\mu_0}\left[-\frac{\partial}{\partial s}\left(\frac{k}{s}\right)\right]\hat{\boldsymbol{\phi}} = \frac{k}{\mu_0 s^2}\,\hat{\boldsymbol{\phi}}$$

Magnetic Vector Potential:

Prob. 5.22 (Griffiths, 3rd Ed.): Find the magnetic vector potential due to a finite wire and using that find the magnetic field.



$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{𝓇}\,d\tau'$$

$$\rightarrow \quad \mathbf{A} = \frac{\mu_0}{4\pi}\int_{-z_1}^{z_2} \frac{I}{𝓇}\,dz\,\hat{\mathbf{z}} \quad = \frac{\mu_0 I}{4\pi}\int_{-z_1}^{z_2} \frac{1}{\sqrt{s^2+z^2}}\,dz\,\hat{\mathbf{z}}$$

$$= \frac{\mu_0 I}{4\pi}\ln\left[\frac{z_2+\sqrt{z_2^2+s^2}}{-z_1+\sqrt{z_1^2+s^2}}\right]\hat{\mathbf{z}}$$

$$\mathbf{B} = \nabla\times\mathbf{A} = -\frac{\partial A_z}{\partial s}\,\hat{\boldsymbol{\phi}}$$

$$= -\frac{\partial}{\partial s}\left(\frac{\mu_0 I}{4\pi}\ln\left[\frac{z_2+\sqrt{z_2^2+s^2}}{-z_1+\sqrt{z_1^2+s^2}}\right]\right)\hat{\boldsymbol{\phi}}$$

$$= \frac{\mu_0 I}{4\pi s}\left(\frac{z_2}{\sqrt{z_2^2+s^2}}+\frac{z_1}{\sqrt{z_1^2+s^2}}\right)\hat{\boldsymbol{\phi}} = \frac{\mu_0 I}{4\pi s}(\sin\theta_2+\sin\theta_1)\,\hat{\boldsymbol{\phi}}$$

Magnetostatics in matter (magnetic field in matter)

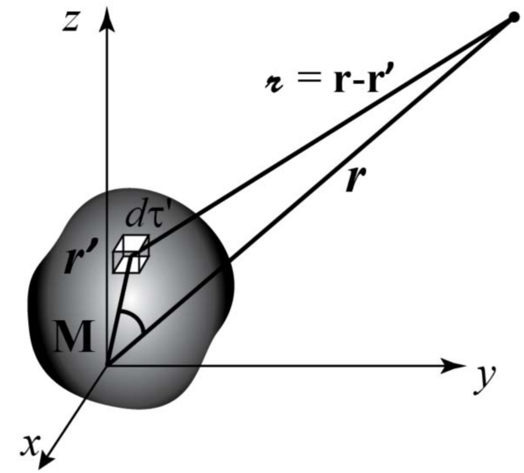
What is Polarization? - dipole moment per unit volume

- The dipole moment is caused either by stretch of an atom/molecule or by rotation of polar molecules
- Polarization in the direction parallel to the applied electric field

What is Magnetization? - magnetic dipole moment per unit volume

- The magnetic dipole moment is caused by electric charges in motion: (i) electrons orbiting around nuclei & (ii) electrons spinning about their own axes.
- In some material, magnetization is in the direction parallel to **B** (Paramagnets).
- In some other material, magnetization is opposite to **B** (Diamagnets).
- In other, there can be magnetization even in the absence of **B** (Ferromagnets).
  - ➢ Magnetization in Ferromagnetic material is much higher.
  - ➢ If a ferromagnetic material is exposed to strong-enough magnetic field, the magnetization in the material does not return to zero after the field is removed and this way the material becomes a permanent magnet.

The Field of a Magnetized Object:



$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\int_{vol}\frac{\mathbf{M}(\mathbf{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

$$= \frac{\mu_0}{4\pi}\int_{vol}\left[\mathbf{M}(\mathbf{r}')\times\boldsymbol{\nabla}'\left(\frac{1}{\mathscr{r}}\right)\right]d\tau'$$

[Using $\boldsymbol{\nabla}'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$]

$$= \frac{\mu_0}{4\pi}\int_{vol}\frac{1}{\mathscr{r}}[\boldsymbol{\nabla}'\times\mathbf{M}(\mathbf{r}')]d\tau' - \frac{\mu_0}{4\pi}\int_{vol}\boldsymbol{\nabla}'\times\left[\frac{\mathbf{M}(\mathbf{r}')}{\mathscr{r}}\right]d\tau'$$

[Using $\boldsymbol{\nabla}\times(f\mathbf{A}) = f(\boldsymbol{\nabla}\times\mathbf{A}) - \mathbf{A}\times(\boldsymbol{\nabla}f)$]

$$= \frac{\mu_0}{4\pi}\int_{vol}\frac{1}{\mathscr{r}}[\boldsymbol{\nabla}'\times\mathbf{M}(\mathbf{r}')]d\tau' + \frac{\mu_0}{4\pi}\oint_{surf}\frac{1}{\mathscr{r}}[\mathbf{M}(\mathbf{r}')\times d\mathbf{a}']$$

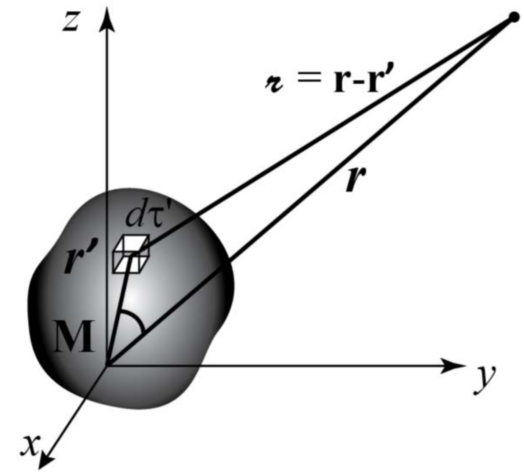
[Griffiths Prob. 1.60 (b) $\int_{vol}(\boldsymbol{\nabla}\times\mathbf{V})d\tau = -\oint_{surf}\mathbf{V}\times d\mathbf{a}$]

$$= \frac{\mu_0}{4\pi}\int_{vol}\frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}d\tau' + \frac{\mu_0}{4\pi}\oint_{surf}\frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}d\mathbf{a}'$$

$\mathbf{J}_b(\mathbf{r}') = \boldsymbol{\nabla}'\times\mathbf{M}(\mathbf{r}')$ Volume current

$\mathbf{K}_b(\mathbf{r}') = \mathbf{M}(\mathbf{r}')\times\hat{\mathbf{n}}$ Surface current

Ampere's law in magnetized material:



$$\mathbf{J}_b(\mathbf{r}') = \nabla' \times \mathbf{M}(\mathbf{r}')$$

Volume current

$$\mathbf{K}_b(\mathbf{r}') = \mathbf{M}(\mathbf{r}') \times \hat{\mathbf{n}}$$

Surface current

Total volume current is

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b$$

What happens to the Ampere's law?

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad = \mu_0(\mathbf{J}_f + \mathbf{J}_b) \quad = \mu_0(\mathbf{J}_f + \nabla \times \mathbf{M})$$

Or, $\nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J}_f$

Define: $\mathbf{H} \equiv \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$ Which means $\mathbf{B} \equiv \mu_0(\mathbf{H} + \mathbf{M})$

So, $\nabla \times \mathbf{H} = \mathbf{J}_f$ Ampere's law in magnetized material (differential form)

And, $\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}}$ Ampere's law in magnetized material (integral form)

Magnetic Susceptibility and Permeability

In paramagnetic and diamagnetic materials, the magnetization is proportional to the applied field

$$\mathbf{M} = \chi_m \, \mathbf{H}$$

- χ_m is called the magnetic susceptibility
- χ_m is a dimensionless quantity
- χ_m is positive for paramagnetic and negative for diamagnets
- Typical values of χ_m are around 10^{-5}

The magnetic field thus becomes

$$\mathbf{B} \equiv \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(\mathbf{H} + \chi_m \mathbf{H}) = \mu_0(1 + \chi_m)\mathbf{H}$$

So, $\mathbf{B} \equiv \mu \mathbf{H}$

$\mu \equiv \mu_0(1 + \chi_m)$
is called the permeability of the material

All the best for midsem