Semester II, 2017-18
Department of Physics, IIT Kanpur

PHY103A: Lecture # 4

(Text Book: Intro to Electrodynamics by Griffiths, 3rd Ed.)

Anand Kumar Jha
10-Jan-2018

Notes

- The Solutions to HW # 1 have been posted.

- HW # 2 has also been posted.

- Office Hour – Friday 2:30-3:30 pm

- I am assuming that everyone has been receiving the course emails.

- Course Webpage: http://home.iitk.ac.in/~akjha/PHY103.htm

Summary of Lecture # 3:

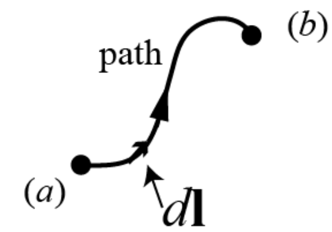
- Line integral, surface integral, and volume integral

- The fundamental Theorem for derivative:

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

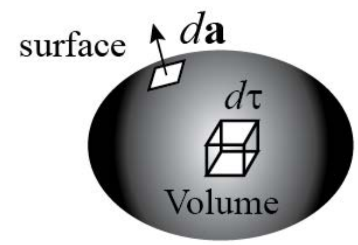
- The fundamental Theorem for Gradient:

$$\int_{a\ Path}^{b} \nabla T \cdot d\mathbf{l} = T(b) - T(a)$$



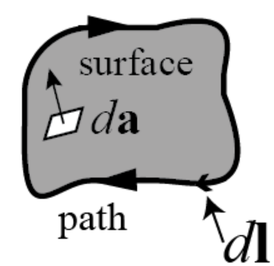
- The fundamental Theorem for Divergence **(Gauss's theorem):**

$$\int_{Vol} (\nabla \cdot \mathbf{V})\,d\tau = \oint_{Surf} \mathbf{V} \cdot d\mathbf{a}$$



- The fundamental Theorem for Curl **(Stokes' theorem):**

$$\int_{Surf} (\nabla \times \mathbf{V}) \cdot d\mathbf{a} = \oint_{Path} \mathbf{V} \cdot d\mathbf{l}$$



- Coordinate systems: Spherical polar, and cylindrical

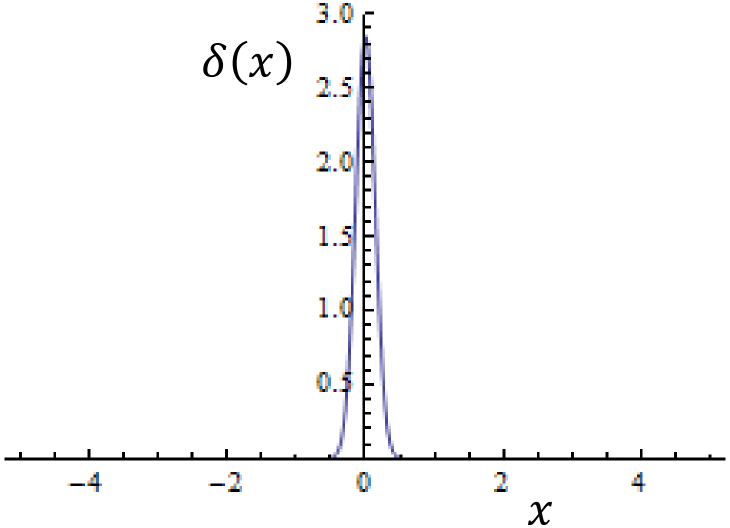
Dirac Delta function:

- Dirac delta function is a special function, which is defined as:

$$\delta(x) = 0, \qquad if\ x \neq 0$$
$$= \infty, \qquad if\ x = 0$$
$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

- Realization of a Dirac Delta function

$$\delta(x) = \lim_{s \to 0} \frac{1}{\sqrt{2\pi s^2}} \exp\left[-\frac{(x-a)^2}{2s^2}\right]$$



- Example: What is the charge density of a point charge q kept at the origin?

$$\rho(x) = q\delta(x); \qquad \int_{-\infty}^{\infty} \rho(x)dx = \int_{-\infty}^{\infty} q\delta(x)dx = q$$

Properties of a Dirac Delta function:

(1) $\delta(kx) = \frac{1}{|k|}\delta(x),$

(3) If $f(x)$ is a continuous function of x

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = \int_{-\infty}^{\infty} f(a)\delta(x-a)dx = f(a)$$

(2) Dirac delta function centered at $x = a$ is defined as follows

$$\begin{aligned}\delta(x-a) &= 0, \qquad if\ x \neq 0 \\ &= \infty, \qquad if\ x = a\end{aligned} \quad \int_{-\infty}^{\infty} \delta(x-a)dx = 1$$

(4) 3D Dirac delta function is defined as:

$$\delta^3(\boldsymbol{r}) = \delta(x)\delta(y)\delta(z) \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\boldsymbol{r})\delta^3(\boldsymbol{r}-\boldsymbol{a}) \ = f(\boldsymbol{a})$$

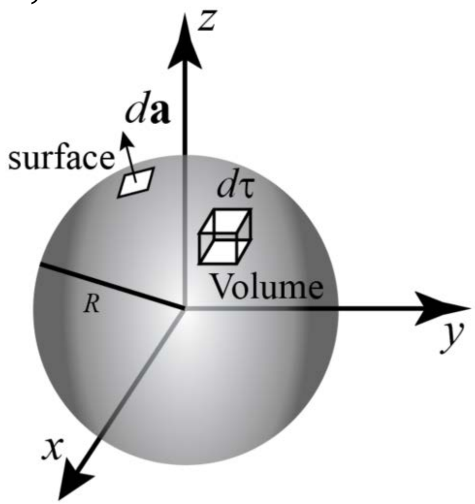
Divergence of the vector field $\mathbf{V} = \frac{\hat{\mathbf{r}}}{r^2}$

Recall Homework Problem 1.5. The divergence $\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^{\mathrm{n}}}\right) = \frac{2-n}{r^{(\mathrm{n}+1)}}$.

So, for $\mathbf{V} = \frac{\hat{\mathbf{r}}}{r^2}$, $\quad \nabla \cdot \mathbf{V} = \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) = \frac{0}{r^3} = 0$ (except at r=0 where it is 0/0, not defined

Let's calculate the divergence using the divergence theorem:

$$\int_{Vol} (\nabla \cdot \mathbf{V}) d\tau = \oint_{Surf} \mathbf{V} \cdot d\mathbf{a}$$



Take the volume integral over a sphere of radius R and the surface integral over the surface of a sphere of radius R.

$$\oint_{Surf} \mathbf{V} \cdot d\mathbf{a} = \iint \left(\frac{\hat{\mathbf{r}}}{R^2}\right) \cdot (R^2 \sin\theta \, d\theta \, d\phi \hat{\mathbf{r}}) = \iint \sin\theta \, d\theta \, d\phi = 4\pi$$

Therefore,

$$\int_{Vol} (\nabla \cdot \mathbf{V}) d\tau = \oint_{Surf} \mathbf{V} \cdot d\mathbf{a} = 4\pi$$

We find that $\nabla \cdot \mathbf{V}$=0 but its integral over a volume is finite. This is possible only if

$$\boxed{\nabla \cdot \mathbf{V} = \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi \, \delta(\mathbf{r})}$$

Divergence of the vector field $\mathbf{V} = \dfrac{\hat{\mathbf{r}}}{r^2}$

Recall Homework Problem 1.5. The divergence $\nabla \cdot \left(\dfrac{\hat{\mathbf{r}}}{r^{n}}\right) = \dfrac{2-n}{r^{(n+1)}}$.

So, for $\mathbf{V} = \dfrac{\hat{\mathbf{r}}}{r^2}$, $\quad \nabla \cdot \mathbf{V} = \nabla \cdot \left(\dfrac{\hat{\mathbf{r}}}{r^2}\right) = \dfrac{0}{r^3} = 0$ (except at r=0 where it is 0/0, not defined)

Let's calculate the divergence using the divergence theorem:

$$\int_{Vol} (\nabla \cdot \mathbf{V}) d\tau = \oint_{Surf} \mathbf{V} \cdot d\mathbf{a}$$

We find that $\nabla \cdot \mathbf{V}$=0 but its integral over a volume is finite. This is only possible if

Take the volume integral over a sph

surface integral over the surface of a

$$\oint_{Surf} \mathbf{V} \cdot d\mathbf{a} = \iint \left(\frac{\hat{\mathbf{r}}}{R^2}\right) \cdot (R^2 \sin\theta$$

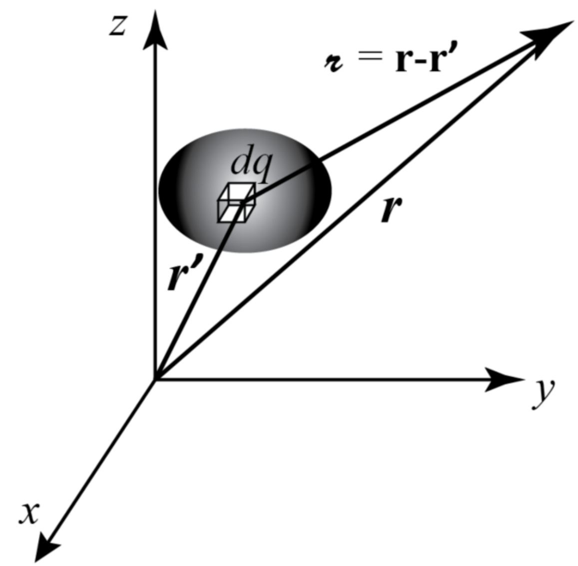
Therefore,

$$\int_{Vol} (\nabla \cdot \mathbf{V}) d\tau = \oint_{Surf} \mathbf{V} \cdot d\mathbf{a} = 4\pi$$

We find that $\nabla \cdot \mathbf{V}$=0 but its integral

$$\boxed{\nabla \cdot \mathbf{V} = \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\, \delta(\mathbf{r})}$$

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) ??$$



$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\, \delta(\boldsymbol{\mathscr{r}}) = 4\pi\, \delta(\mathbf{r} - \mathbf{r}')$$

A few more essential concepts:

1. The Helmholtz theorem

A vector field **F** in elctrodynamics can be completely determined if:

(i) The divergence $\nabla \cdot \mathbf{F}$ is known
(ii) The curl $\nabla \times \mathbf{F}$ is known
(iii) If the field goes to zero at infinity.

For the proof of this theorem, see Appendix B of Griffiths

A few more essential concepts:

2. Scalar Potential:

If the curl of a vector field $\mathbf{F}$ is zero, that is, if $\nabla \times \mathbf{F} = 0$ everywhere, then:

(1) $\int_a^b \mathbf{F} \cdot d\mathbf{l}$ is independent of path.

(2) $\oint \mathbf{F} \cdot d\mathbf{l} = 0$ for any closed loop.

This is because of Stokes' theorem

$$\int_{Surf} (\nabla \times \mathbf{F}) \cdot d\mathbf{a} = \oint_{Path} \mathbf{F} \cdot d\mathbf{l}$$

(3) $\mathbf{F}$ is the gradient of a scalar function: $\mathbf{F} = -\nabla \mathrm{V}$

- This is because Curl of a gradient is zero $\nabla \times (\nabla \mathrm{V}) = \mathbf{0}$
- The minus sign is purely conventional.
- The scalar potential not unique. A constant can be added to V without affecting the gradient: $\nabla \mathrm{V} = \nabla(\mathrm{V} + \mathrm{a})$, since the gradient of a constant is zero.

A few more essential concepts:

3. Vector Potential:

If the divergence of a vector field $\mathbf{F}$ is zero, that is, if $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$ everywhere, then:

(1) $\int \mathbf{F} \cdot d\mathbf{a}$ is independent of surface.

(2) $\oint \mathbf{F} \cdot d\mathbf{a} = 0$ for any closed surface.

This is because of the divergence theorem

$$\int_{Vol} (\boldsymbol{\nabla} \cdot \mathbf{F}) d\tau = \oint_{Surf} \mathbf{F} \cdot d\mathbf{a}$$

(3) $\mathbf{F}$ is the gradient of a vector function: $\mathbf{F} = \boldsymbol{\nabla} \times \mathbf{A}$

- This is because divergence of a curl is zero $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = 0$
- The vector potential is not unique. A gradient $\boldsymbol{\nabla}\mathrm{V}$ of a scalar function can be added to $\mathbf{A}$ without affecting the curl, since the curl of a gradient is zero.

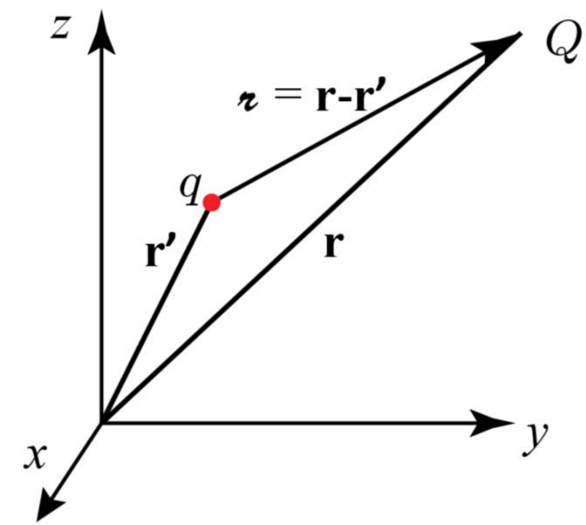
Electric Field

Coulomb's Law

Force on a test charge Q due to a single point charge q is:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q\,Q}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}} \quad \text{(in the units of Newton)}$$

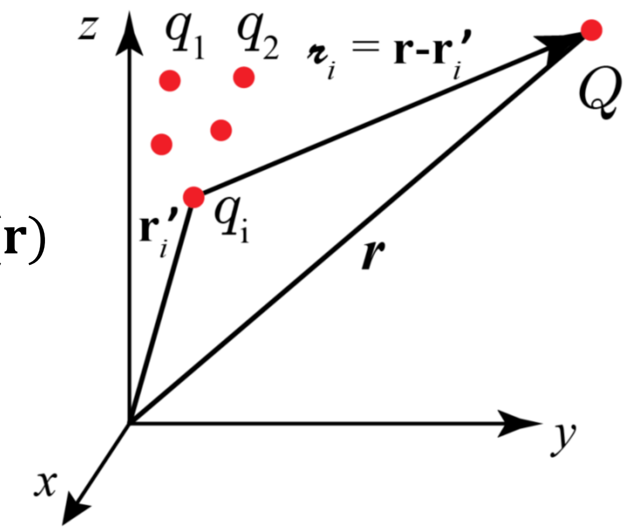
$$\epsilon_0 = 8.85 \times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2}$$ Permittivity of free space



Force on a test charge Q due to a collection of point charges is:

$$\mathbf{F}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1\,Q}{\mathscr{r}_1^{\,2}}\,\widehat{\boldsymbol{\mathscr{r}}_1} + \frac{q_2\,Q}{\mathscr{r}_2^{\,2}}\,\widehat{\boldsymbol{\mathscr{r}}_2} + \frac{q_3\,Q}{\mathscr{r}_3^{\,2}}\,\widehat{\boldsymbol{\mathscr{r}}_3} + \cdots\right) = Q\mathbf{E}(\mathbf{r})$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{\mathscr{r}_1^{\,2}}\,\widehat{\boldsymbol{\mathscr{r}}_1} + \frac{q_2}{\mathscr{r}_2^{\,2}}\,\widehat{\boldsymbol{\mathscr{r}}_2} + \frac{q_3}{\mathscr{r}_3^{\,2}}\,\widehat{\boldsymbol{\mathscr{r}}_3} + \cdots\right)$$



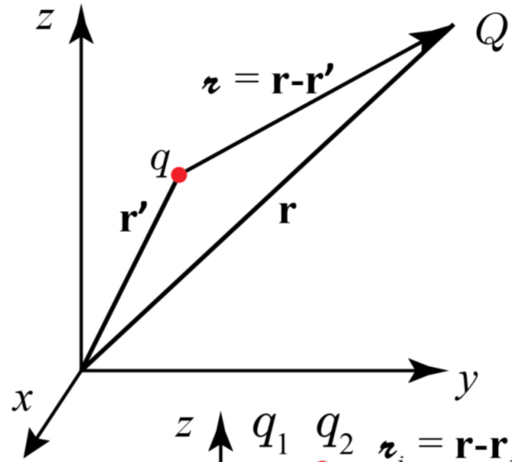
Q: What is field (a vector function), physically?

A: We don't really know. At this level, field is just a mathematical concept which is consistent with the physical theory (electrodynamics). Also, we know how to calculate a field.

Electric Field

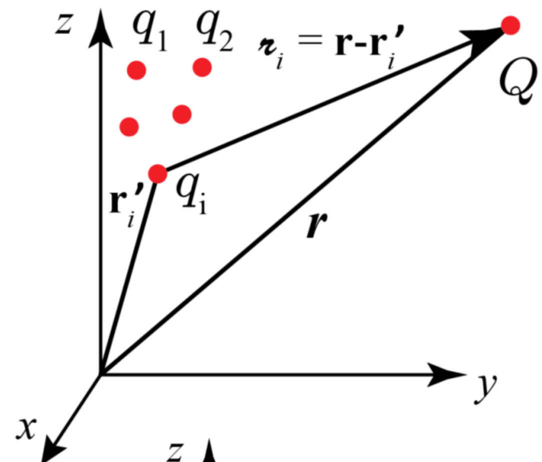
Electric field due to a single point charge q is:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}$$



Electric field due to a collection of point charges is:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{\mathscr{r}_1{}^2}\,\widehat{\boldsymbol{\mathscr{r}}_1} + \frac{q_2}{\mathscr{r}_2{}^2}\,\widehat{\boldsymbol{\mathscr{r}}_2} + \frac{q_3}{\mathscr{r}_3{}^2}\,\widehat{\boldsymbol{\mathscr{r}}_3} + \cdots\right)$$



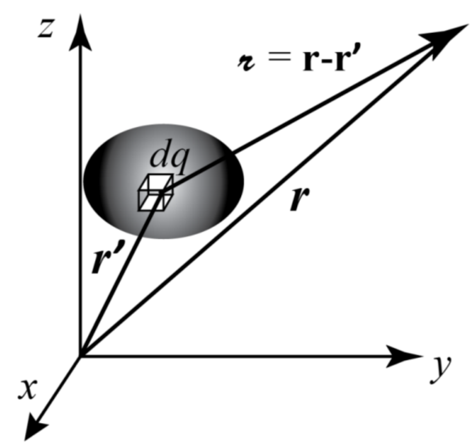
Electric field due to a continuous charge distribution is:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{dq}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}$$

For a line charge $dq = \lambda(\mathbf{r}')dl$

For a surface charge $dq = \sigma(\mathbf{r}')da$

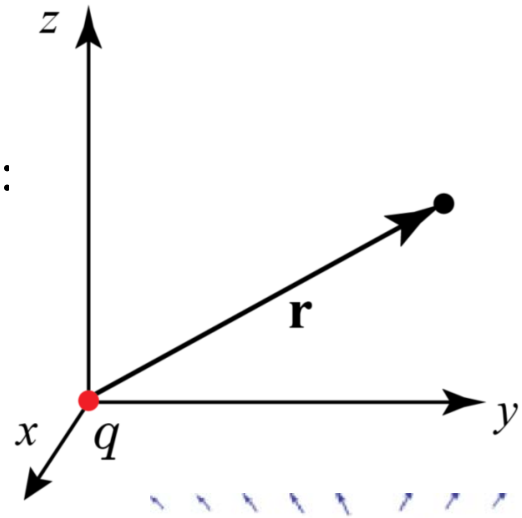
For a volume charge $dq = \rho(\mathbf{r}')d\tau$



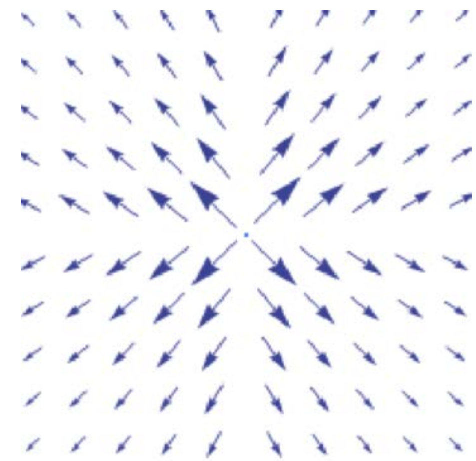
We know how to calculate electric fields due a charge distribution, using Coulomb’s law. We’ll now explore some tricks for calculating the field more efficiently.

Electric Flux and Gauss's Law:

Electric field $\mathbf{E}(\mathbf{r})$ due to a single point charge q at origin is:

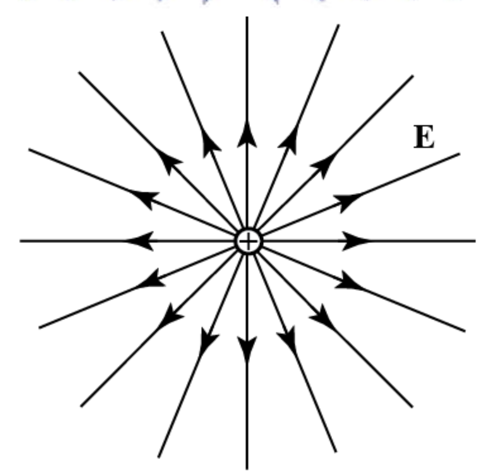


$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}}$$



Electric field is a vector field. One way to represent a vector field is by drawing a vectors of given magnitude and directions.

Another way to represent a vector field is by drawing the field lines:



(i) Field lines emanate from the positive charge and end up on the negative charge or go up to infinity.

(ii) The density is the filed lines is proportional to the strength of the field.

Electric Flux and Gauss's Law:

Electric field $\mathbf{E}(\mathbf{r})$ due to a single point charge q at origin is:

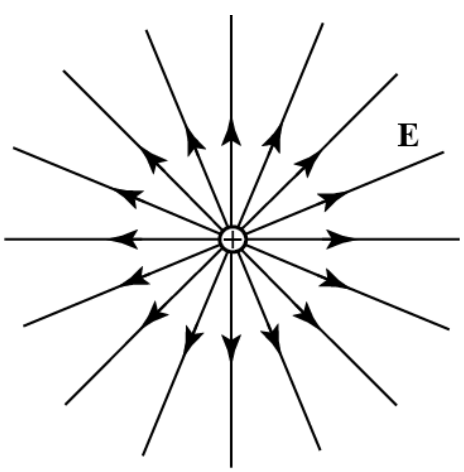
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}}$$

The electric flux is defined as

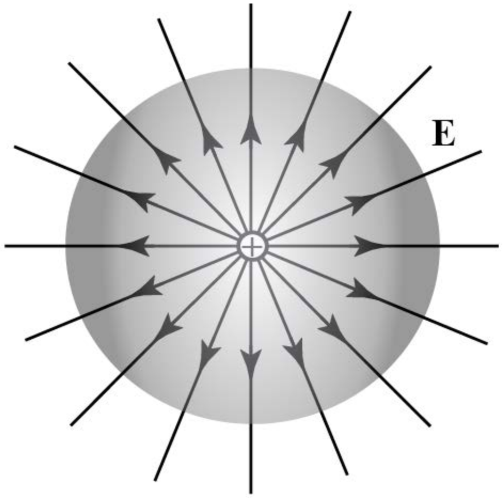
$$\Phi_E = \oint_{surf} \mathbf{E} \cdot d\mathbf{a}$$

$\mathbf{E} \cdot d\mathbf{a}$ is proportional the number of field lines passing through an area element $d\mathbf{a}$

When the area $d\mathbf{a}$ is perpendicular to the field $\mathbf{E}$ the dot product is zero.



Electric Flux and Gauss's Law:



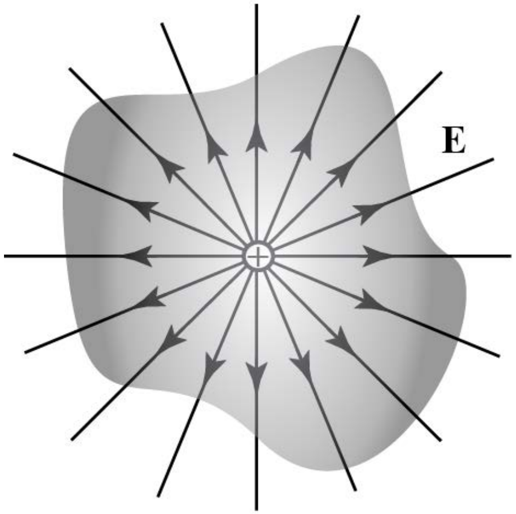
Electric field $\mathbf{E}(\mathbf{r})$ due to a single point charge q at origin is:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\ \hat{\mathbf{r}}$$

The electric flux due to a point charge q at origin through a spherical shell of radius R.

$$\Phi_E = \oint_{surf} \mathbf{E} \cdot d\mathbf{a} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\ \hat{\mathbf{r}} \cdot R^2 \sin\theta d\theta d\phi \hat{\mathbf{r}} = \frac{q}{\epsilon_0}$$

Electric Flux and Gauss's Law:



Electric field $\mathbf{E}(\mathbf{r})$ due to a single point charge q at origin is:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\ \hat{\mathbf{r}}$$

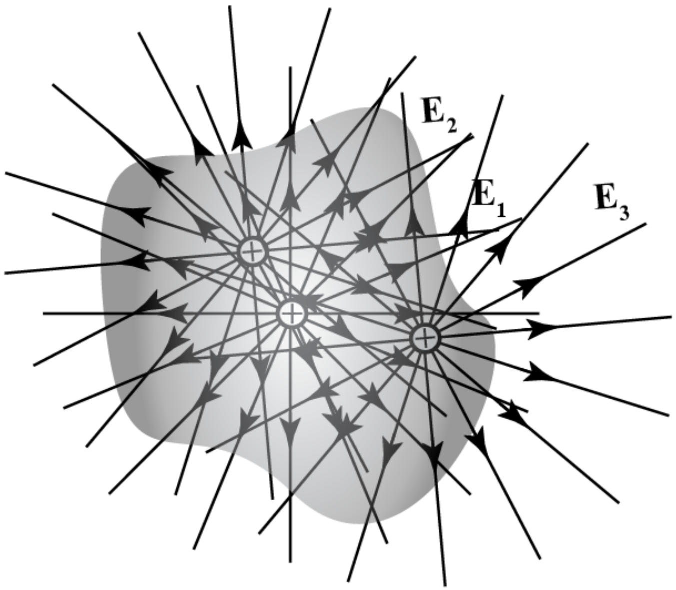
The electric flux due to a point charge q at origin through any enclosing closed surface is

$$\Phi_E = \oint_{surf} \mathbf{E} \cdot d\mathbf{a} = \frac{q}{\epsilon_0}$$

Electric Flux and Gauss's Law:

Electric field $\mathbf{E}(\mathbf{r})$ due to a single point charge q at origin is:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}}$$



The electric flux due to a collection of point charges through any enclosing closed surface.

$$\Phi_E = \oint_{surf} \mathbf{E}\cdot d\mathbf{a} = ??$$

$$= \oint_{surf} \left(\sum_{i=1}^{n} \mathbf{E}_i\right) . d\mathbf{a} = \sum_{i=1}^{n} \oint_{surf} \mathbf{E}_i . d\mathbf{a} = \sum_{i=1}^{n} \frac{q_i}{\epsilon_0}$$

$$\boxed{\oint_{surf} \mathbf{E}\cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}}$$

This is the Gauss's law in the integral form: The flux through a surface is equal to the total charge enclosed by the surface divided by ϵ_0

Electric Flux and Gauss's Law:

$$\boxed{\oint_{surf} \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}}$$

This is the Gauss's law in integral form.

Or, $\displaystyle\int_{Vol} (\nabla \cdot \mathbf{E}) d\tau = \frac{Q_{\text{enc}}}{\epsilon_0}$

Using divergence theorem $\displaystyle\int_{Vol} (\nabla \cdot \mathbf{E}) d\tau = \oint_{Surf} \mathbf{E} \cdot d\mathbf{a}$

Or, $\displaystyle\int_{Vol} (\nabla \cdot \mathbf{E}) d\tau = \int \frac{1}{\epsilon_0} \rho(\mathbf{r}') d\tau$

For a volume Charge density $\displaystyle Q_{\text{enc}} = \int \rho(\mathbf{r}') d\tau$

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}}$$

This is the Gauss's law in differential form

Gauss's law doesn't have any information that the Coulomb's does not contain. The importance of Gauss's law is that it makes calculating electric field much simpler and provide a deeper understanding of the field itself.

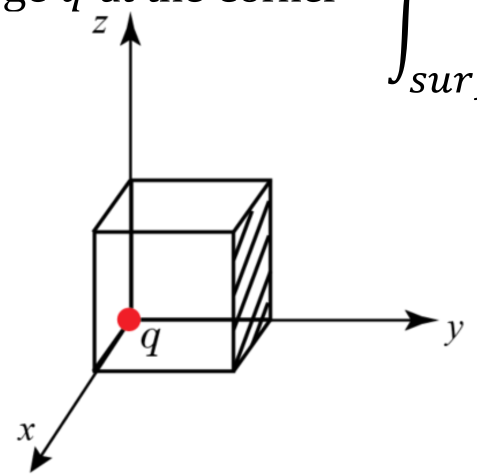
Electric Flux and Gauss's Law:

$$\boxed{\oint_{surf} \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}}$$

This is the Gauss's law in integral form.

Q: (Griffiths: Ex 2.10): What is the flux through the shaded face of the cube due to the charge q at the corner

$$\int_{surf} \mathbf{E} \cdot d\mathbf{a} \; ??$$



Answer:

$$24 \int_{surf} \mathbf{E} \cdot d\mathbf{a} = \frac{q}{\epsilon_0}$$

$$\int_{surf} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{24} \frac{q}{\epsilon_0}$$