Semester II, 2017-18
Department of Physics, IIT Kanpur

PHY103A: Lecture # 5

(Text Book: Intro to Electrodynamics by Griffiths, 3rd Ed.)

Anand Kumar Jha
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Summary of Lecture # 4:

- Scalar Potential : if $\nabla \times \mathbf{F} = 0$ everywhere, $\mathbf{F} = -\nabla V$

- Vector Potential : if $\nabla \cdot \mathbf{F} = 0$ everywhere, $\mathbf{F} = \nabla \times \mathbf{A}$

- Coulomb's Law:
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{\imath^2} \hat{\boldsymbol{\imath}}$$

- Electric Flux
$$\Phi_E = \oint_{surf} \mathbf{E} \cdot d\mathbf{a}$$

- Gauss's Law
$$\oint_{surf} \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{enc}}{\epsilon_0} \qquad \text{(in integral form)}$$
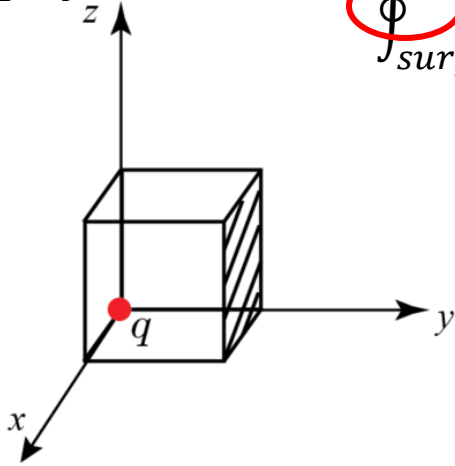
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \text{(in differential form)}$$

Correction in Lecture # 4:

$$\oint_{surf} \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{enc}}{\epsilon_0}$$

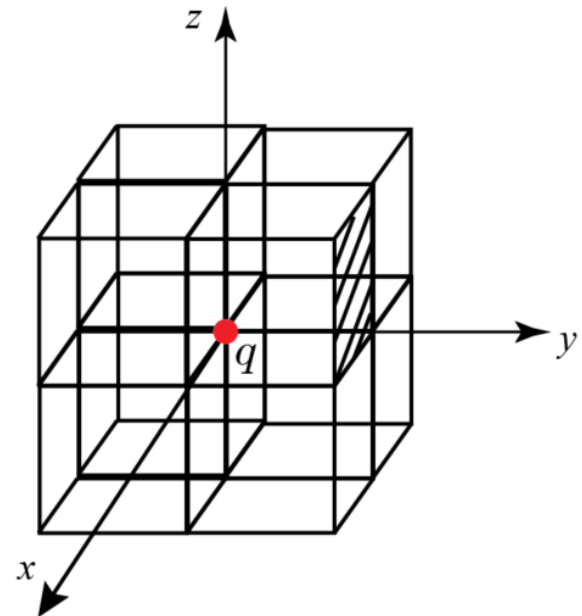This is the Gauss's law in integral form.

Q: (Griffiths: Ex 2.10): What is the flux through the shaded face of the cube due to the charge q at the corner $\oint_{surf} \mathbf{E} \cdot d\mathbf{a}$? ?

Answer:

$$24 \oint_{surf} \mathbf{E} \cdot d\mathbf{a} = \frac{q}{\epsilon_0}$$

$$\oint_{surf} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{24} \frac{q}{\epsilon_0}$$

Gauss's Law from Coulomb's Law:

If Coulomb's Law and Gauss's law have the same information content, can we derive Gauss's law from Coulomb's law?

Coulomb's law gives the electric field due to a volume charge $\rho(\mathbf{r}')$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\hat{\boldsymbol{\imath}}}{\imath^2} \, \rho(\mathbf{r}')d\tau$$
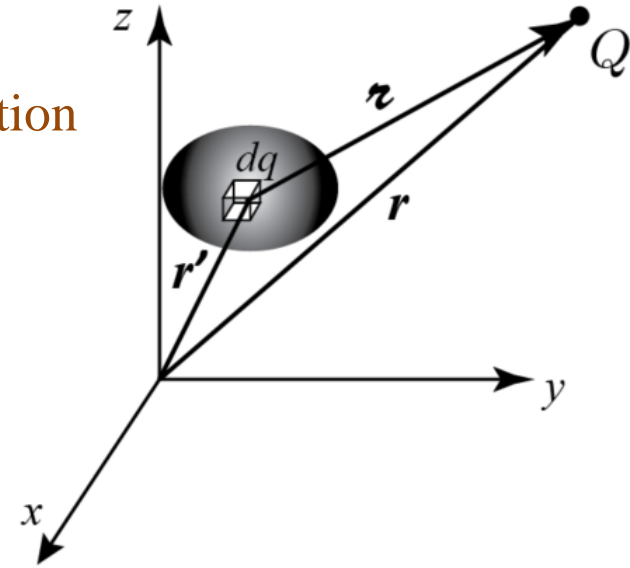
Take the divergence of both sides of the equation

$$\boldsymbol{\nabla} \cdot \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \boldsymbol{\nabla} \cdot \left(\frac{\hat{\boldsymbol{\imath}}}{\imath^2}\right) \rho(\mathbf{r}')d\tau$$

We have: $\boldsymbol{\nabla} \cdot \left(\dfrac{\hat{\boldsymbol{\imath}}}{\imath^2}\right) = 4\pi \, \delta(\boldsymbol{\imath}) = 4\pi \, \delta(\mathbf{r} - \mathbf{r}')$

Therefore, $\boldsymbol{\nabla} \cdot \mathbf{E}(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0} \int 4\pi \, \delta(\mathbf{r} - \mathbf{r}') \, \rho(\mathbf{r}')d\tau = \dfrac{\rho(\mathbf{r})}{\epsilon_0}$

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}}$$

The divergence of electric field is equal to the charge density divided by ϵ_0

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Curl of the Electric Field:

Let's take the simplest electric field:

Electric field due to a single point charge q is:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

We need to find the curl of it

$$\boldsymbol{\nabla}\times\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\boldsymbol{\nabla}\times\left(\frac{1}{r^2}\hat{\mathbf{r}}\right)$$

Take the area integral

$$\int_{surf}\boldsymbol{\nabla}\times\mathbf{E}(\mathbf{r})\cdot d\mathbf{a} = \frac{q}{4\pi\epsilon_0}\int_{surf}\boldsymbol{\nabla}\times\left(\frac{1}{r^2}\hat{\mathbf{r}}\right)\cdot d\mathbf{a}$$
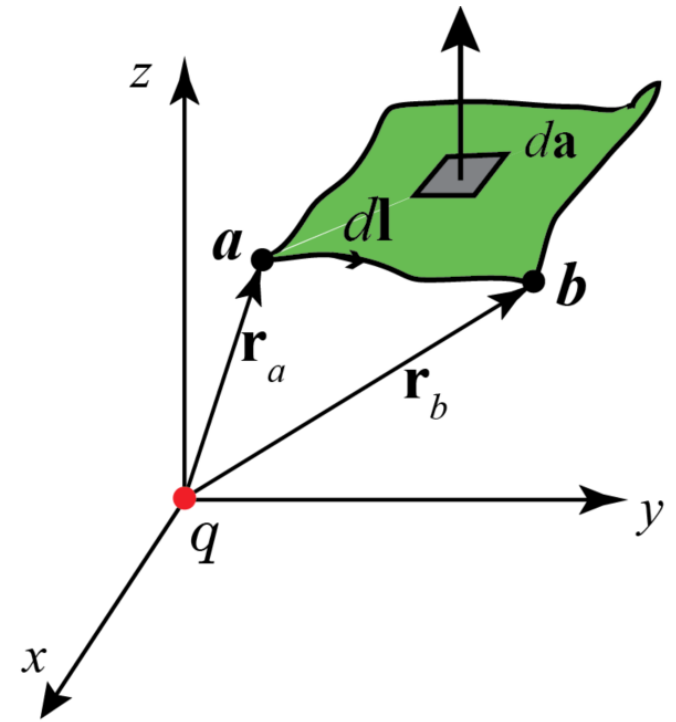
Use Stokes's theorem

$$\int_{surf}\boldsymbol{\nabla}\times\mathbf{E}(\mathbf{r})\cdot d\mathbf{a} = \frac{q}{4\pi\epsilon_0}\oint\left(\frac{1}{r^2}\hat{\mathbf{r}}\right)\cdot d\mathbf{l} = \frac{q}{4\pi\epsilon_0}\oint\frac{1}{r^2}dr$$

since
$$d\mathbf{l} = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} \\ +r\sin\theta d\phi\hat{\boldsymbol{\phi}}$$

$$\int_{surf}\boldsymbol{\nabla}\times\mathbf{E}(\mathbf{r})\cdot d\mathbf{a} = \left.\frac{q}{4\pi\epsilon_0}\times\frac{1}{r}\right|_{r_a}^{r_a} = 0 \quad \textbf{implies} \qquad \boxed{\boldsymbol{\nabla}\times\mathbf{E} = \mathbf{0}}$$

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Curl of the Electric Field (Digression):

$$\nabla \times \mathbf{E} = 0$$

Curl of an electric field is zero. We have shown this for the simplest field, which is the field of a point charge. But it can be shown to be true for any electric field, as long as the field is static.

What if the field is dynamic, that is, what if the field changes as a function of time?

$$\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}$$ Faraday's Law in differential form.

Integrate over a surface

$$\int_{surf} (\nabla \times \mathbf{E}) \cdot d\mathbf{a} = \int_{surf} -\frac{d\mathbf{B}}{dt} \cdot d\mathbf{a}$$

Apply Stokes' theorem

$$\int_{path} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_{surf} \mathbf{B} \cdot d\mathbf{a}$$

EMF ↗ $$\boxed{\mathcal{E} = -\frac{d\Phi}{dt}}$$ ← Magnetic flux   Faraday's Law in integral form.

Maxwell's Equations (Digression 2):

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \longleftrightarrow \qquad \oint_{surf} \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0} \qquad \text{Gauss's Law}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{d\mathbf{B}}{dt} \quad \longleftrightarrow \quad \int_{path} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_{\boldsymbol{surf}} \mathbf{B} \cdot d\mathbf{a} \quad \text{Faraday's Law}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \qquad\qquad\qquad\qquad \text{No name; Magnetic Monopole does not exist}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} - \mu_0 \epsilon_0 \frac{d\mathbf{E}}{dt} \qquad \text{Amperes's Law with Maxwell's correction}$$

When fields do not vary as a function of time, it is called Electrostatics / Magnetostatics. (before mid-sem)

When fields do vary as a function of time, then the two fields have to be studied together as an electromagnetic field, and one consequence of a changing electric and magnetic field is the electromagnetic radiation. (after mid-sem)

When the energy of the field is quantized (photons) then it is called quantum electrodynamics. (Not for this course). Applications: Quantum computers, Quantum cryptography, Quantum teleportation

Electric Potential:

Recall: If the curl of a vector field **F** is zero, that is, if $\nabla \times \mathbf{F} = 0$ everywhere, then:

(1) $\int_a^b \mathbf{F} \cdot d\mathbf{l}$ is independent of path.

This is because of Stokes' theorem

(2) $\oint \mathbf{F} \cdot d\mathbf{l} = 0$ for any closed loop.

$$\int_{Surf} (\nabla \times \mathbf{F}) \cdot d\mathbf{a} = \oint_{Path} \mathbf{F} \cdot d\mathbf{l}$$

(3) **F** is the gradient of a scalar function: $\mathbf{F} = -\nabla V$

- This is because Curl of a gradient is zero $\nabla \times (\nabla V) = \mathbf{0}$

The curl of Electric field **E** is zero, that is, $\nabla \times \mathbf{E} = 0$ everywhere. Therefore:

(1) $\int_a^b \mathbf{E} \cdot d\mathbf{l}$ is independent of path.

This is because of Stokes' theorem

(2) $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ for any closed loop.

$$\int_{Surf} (\nabla \times \mathbf{E}) \cdot d\mathbf{a} = \oint_{Path} \mathbf{E} \cdot d\mathbf{l}$$

(3) **E** is the gradient of a scalar function: $\boxed{\mathbf{E} = -\nabla V}$

V is called the electric potential. It is a scalar quantity, the gradient of which is equal to the electric field
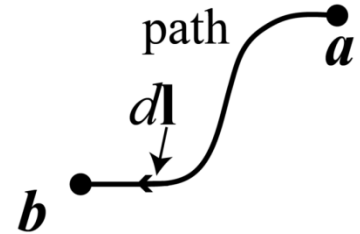
Electric Potential:

Since $\boldsymbol{\nabla} \times \mathbf{E} = 0$ everywhere, $\mathbf{E} = -\boldsymbol{\nabla}V$

How to write electric potential in terms of the electric field?

$$-\boldsymbol{\nabla}V = \mathbf{E}$$

Take the line integral of the above equation over a path

$$\int_a^b \boldsymbol{\nabla}V \cdot d\mathbf{l} = -\int_a^b \mathbf{E} \cdot d\mathbf{l}$$

Use the fundamental Theorem for Gradient: $\int_{a\ Path}^b \boldsymbol{\nabla}V \cdot d\mathbf{l} = V(b) - V(a)$

$$\boxed{V(\mathbf{b}) - V(\mathbf{a}) = -\int_a^b \mathbf{E} \cdot d\mathbf{l}}$$

- Absolute potential cannot be defined.
- Only potential differences can be defined.

Electric Potential:

(1) Electric potential is different from electric potential energy. Unit of electric potential is Newton-meter per Coulomb ($\frac{N \cdot m}{C}$) or Volt.

(2) The potential obeys superposition principle, i.e., the potential due to several charges is equal to the sum of the potentials due to individual ones: $V = V_1 + V_2 + \cdots$

(3) If one knows the electrical potential (a scalar quantity), the electric field (a vector quantity) can be calculated

(4) The electric field is a vector quantity, but we still get all the information from the potential (a scalar quantity). This is because different components are interrelated: $\mathbf{\nabla} \times \mathbf{E} = 0$, i.e., $\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}$; $\qquad \frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z}$; $\qquad \frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}$;

(5) $V(\mathbf{b}) - V(\mathbf{a}) = -\int_a^b \mathbf{E} \cdot d\mathbf{l}$. Absolute potential cannot be defined. In electrostatics, usually one takes the reference point to infinity and set the potential at infinity to zero, that is, take $V(\mathbf{a}) = V(\infty) = 0$. Also if $V(\mathbf{b}) = V(\mathbf{r})$,

$$V(\mathbf{r}) = -\int_\infty^r \mathbf{E} \cdot d\mathbf{l}$$

path

$d\mathbf{l}$

∞

$\mathbf{r}$

Electric Potential due to a point charge at origin:

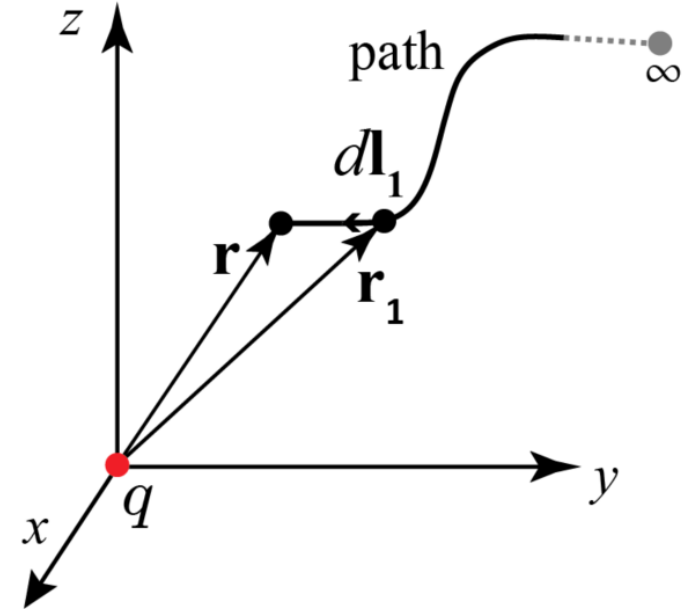Electric field $\mathbf{E}(\mathbf{r_1})$ at $\mathbf{r_1}$ due to a single point charge q at origin:

$$\mathbf{E}(\mathbf{r_1}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r_1^2}\ \hat{\mathbf{r_1}}$$

Electric potential $V(\mathbf{r})$ at $\mathbf{r}$ due to a single point charge q at origin:

$$V(\mathbf{r}) = -\int_\infty^r \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^r \mathbf{E}(\mathbf{r_1})\cdot d\mathbf{l_1}$$

The line element is: $d\mathbf{l_1} = dr_1\ \hat{\mathbf{r_1}} + r_1 d\theta_1\ \hat{\boldsymbol{\theta_1}} + r_1\sin\theta_1 d\phi_1\ \hat{\boldsymbol{\phi_1}}$

$$V(\mathbf{r}) = -\int_\infty^r \frac{1}{4\pi\epsilon_0}\frac{q}{r_1^2}dr_1 = -\frac{q}{4\pi\epsilon_0}\int_\infty^r \frac{1}{r_1^2}dr_1 = \frac{q}{4\pi\epsilon_0}\frac{1}{r}$$

$$\boxed{V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{r}}$$

Electric Potential due to localized charge distribution:

Potential due to a point charge q at origin:

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{r}$$

Potential due to a point charge q at $\mathbf{r}'$:

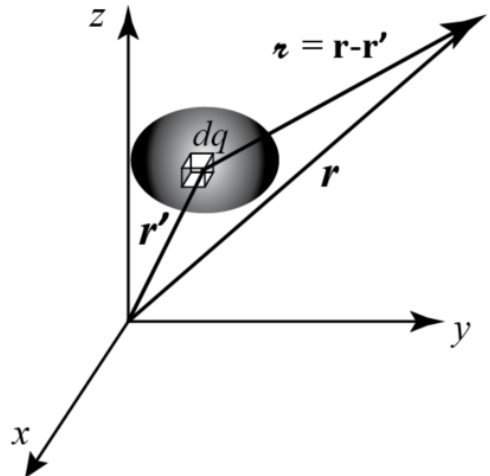$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{\imath}$$
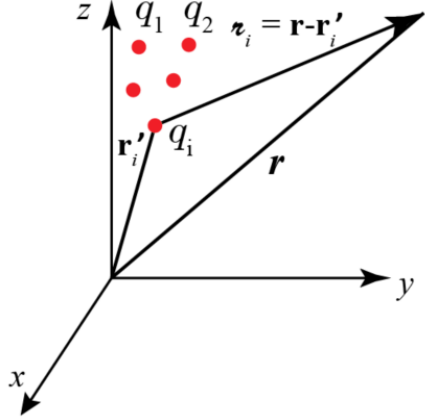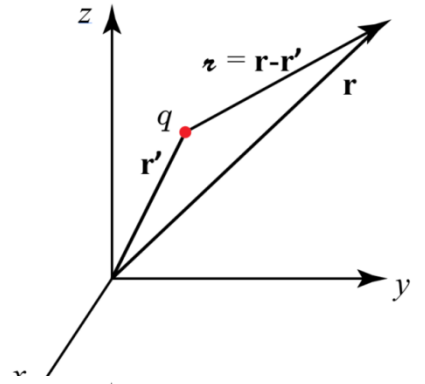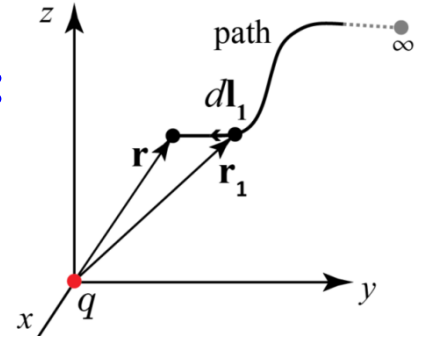
Potential due to a collection of point charges

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\imath_i}$$

Potential due to a a continuous charge distribution is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{dq}{\imath}$$

For a line charge $dq = \lambda(\mathbf{r}')dl'$
For a surface charge $dq = \sigma(\mathbf{r}')da'$
For a volume charge $dq = \rho(\mathbf{r}')d\tau'$

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Ease of calculating the Electric Field

- The easiest way to calculate the electric field is using Gauss's law. But this is possible only when there is some symmetry in the problem.

- The next best thing: if the electric potential is known, one can calculate the electric field by just taking the gradient of the potential $\mathbf{E} = -\boldsymbol{\nabla}V$. Sometimes, it is very effective to calculate the electric potential first and then the electric field from there.

- If the above two is not applicable, one has to go back to the Coulomb's law and then calculate the electric field.