Semester II, 2017-18
Department of Physics, IIT Kanpur

PHY103A: Lecture # 6

(Text Book: Intro to Electrodynamics by Griffiths, 3rd Ed.)

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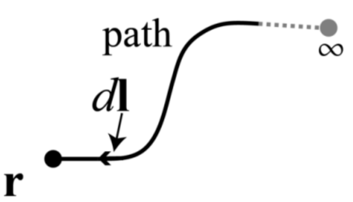
Summary of Lecture # 5:

- Gauss's Law from Coulomb's Law: $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$

- Curl of the electric field : $\nabla \times \mathbf{E} = 0$

- Electric Potential: $\mathbf{E} = -\nabla V \longleftrightarrow V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\boldsymbol{a}}^{\boldsymbol{b}} \mathbf{E} \cdot d\mathbf{l}$

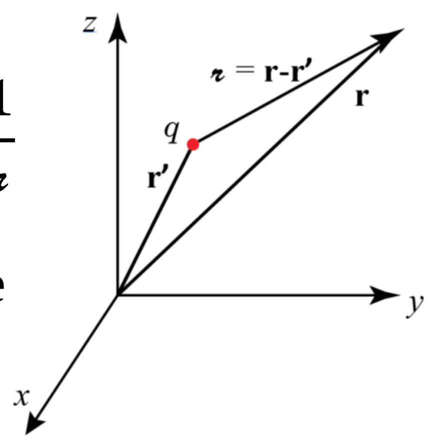
$$V(\mathbf{r}) = -\int_{\infty}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$$



- Electric Potential due to a localized charge distribution

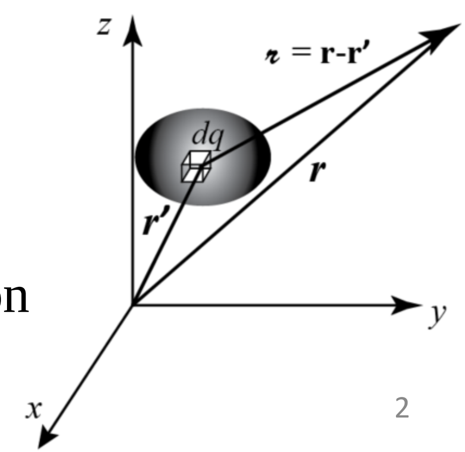
$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}}$$

Point charge



$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}}$$

Charge distribution



Charge distribution in terms of electric potential:

Gauss's Law $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$

But $\nabla \times \mathbf{E} = 0 \quad \Rightarrow \mathbf{E} = -\nabla \mathrm{V}$

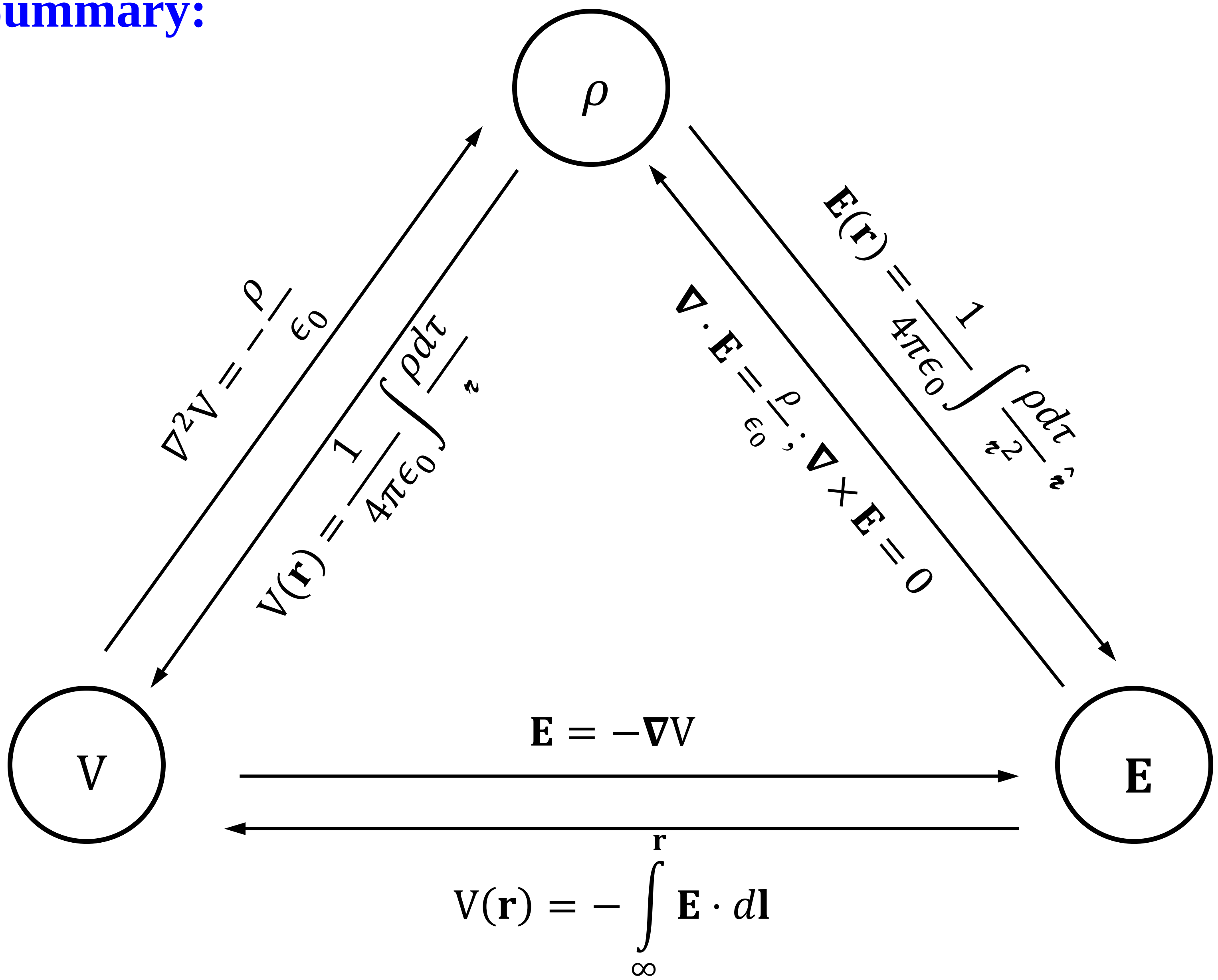
Therefore, $\nabla \cdot \mathbf{E} = \nabla \cdot (-\nabla \mathrm{V}) = -\nabla \cdot (\nabla \mathrm{V}) = -\nabla^2 \mathrm{V} = \frac{\rho}{\epsilon_0}$

$$\boxed{\nabla^2 \mathrm{V} = -\frac{\rho}{\epsilon_0}}$$ **Poisson's Equation**

In the region of space where there is no charge, $\rho=0$

$$\boxed{\nabla^2 \mathrm{V} = 0}$$ **Laplace's Equation**

Summary:

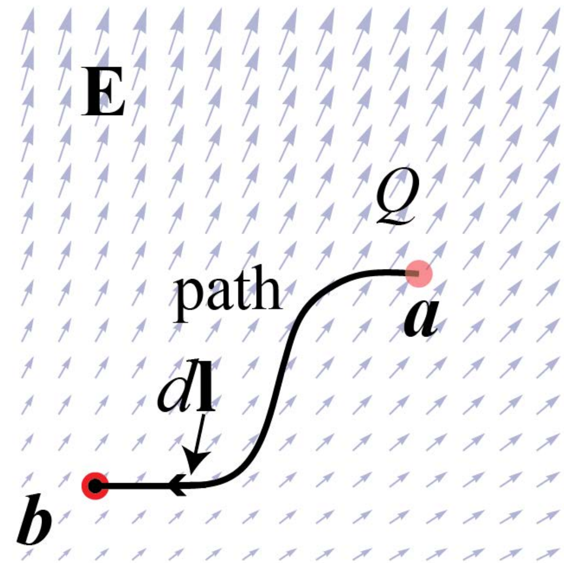


Electrostatics

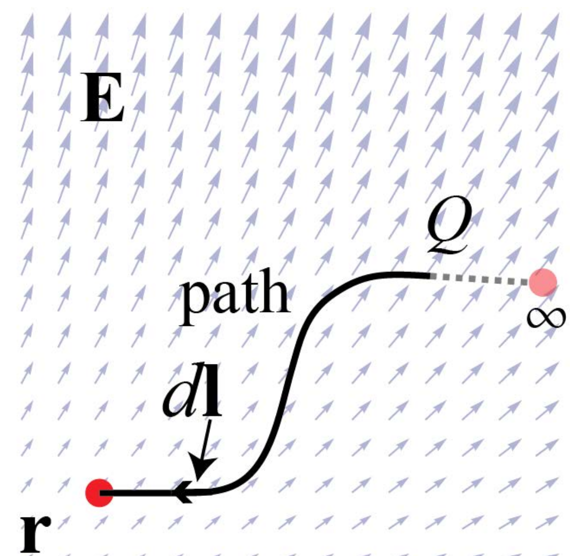
Work and Energy in Electrostatics

There is a charge Q in an electrostatic field $\mathbf{E}$. How much work needs to be done in order to move the charge from point $\boldsymbol{a}$ to $\boldsymbol{b}$?

$$W = \int_{a}^{b} \mathbf{F} \cdot d\mathbf{l} = -Q \int_{a}^{b} \mathbf{E} \cdot d\mathbf{l} = \mathrm{Q}[\mathrm{V}(\mathbf{b}) - \mathrm{V}(\mathbf{a})]$$



- $\mathbf{F} = -Q\mathbf{E}$ is the force one has to exert in order to counteract the electrostatic force $\mathbf{F} = Q\mathbf{E}$.
- Work done to move a unit charge from point $\boldsymbol{a}$ to $\boldsymbol{b}$ is the potential difference between points $\boldsymbol{b}$ and $\boldsymbol{a}$
- Work is independent of the path.



Take $\mathrm{V}(\mathbf{a})=\mathrm{V}(\infty)=0$ and $\mathrm{V}(\mathbf{b}) = \mathrm{V}(\mathbf{r})$

$$W = \mathrm{QV}(\mathbf{r})$$

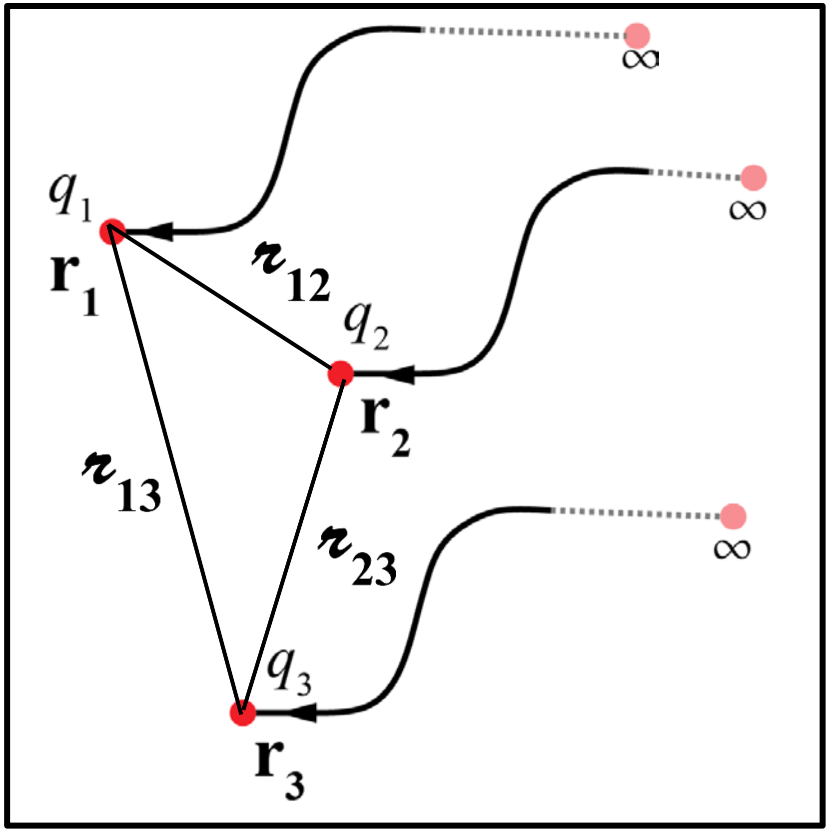
If $\mathrm{Q} = 1$,

$$\boxed{W = \mathrm{V}(\mathbf{r})}$$

- Work done to construct a system of unit charge (to bring a unit charge from ∞ to **r** is the electric potential.
- Thus, electric potential is the potential energy per unit charge

Work required to assemble n point charges:

The work required to construct a system of one point charge Q is: $W = QV(\mathbf{r})$.



Work required to bring in the charge q_1 from ∞ to $\mathbf{r_1}$: $W_1 = q_1 V_0 = q_1 \times 0 = 0$

Work required to bring in the charge q_2 from ∞ to $\mathbf{r_2}$: $W_2 = q_2 V_1 = q_2 \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{\mathscr{r}_{12}}\right)$

Work required to bring in the charge q_3 from ∞ to $\mathbf{r_3}$: $W_3 = q_3 V_2 = q_3 \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{\mathscr{r}_{13}} + \frac{q_2}{\mathscr{r}_{23}}\right)$

Total work required to bring in the first three charges:

$$W = W_1 + W_2 + W_3 = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{q_1 q_3}{\mathscr{r}_{13}} + \frac{q_2 q_3}{\mathscr{r}_{23}}\right)$$

Total work required to bring in the first four charges:

$$W = W_1 + W_2 + W_3 + \mathrm{W}_4 = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{q_1 q_3}{\mathscr{r}_{13}} + \frac{q_2 q_3}{\mathscr{r}_{23}} + \frac{q_1 q_4}{\mathscr{r}_{14}} + \frac{q_2 q_4}{\mathscr{r}_{24}} + \frac{q_3 q_4}{\mathscr{r}_{34}}\right)$$

Work required to assemble n point charges:

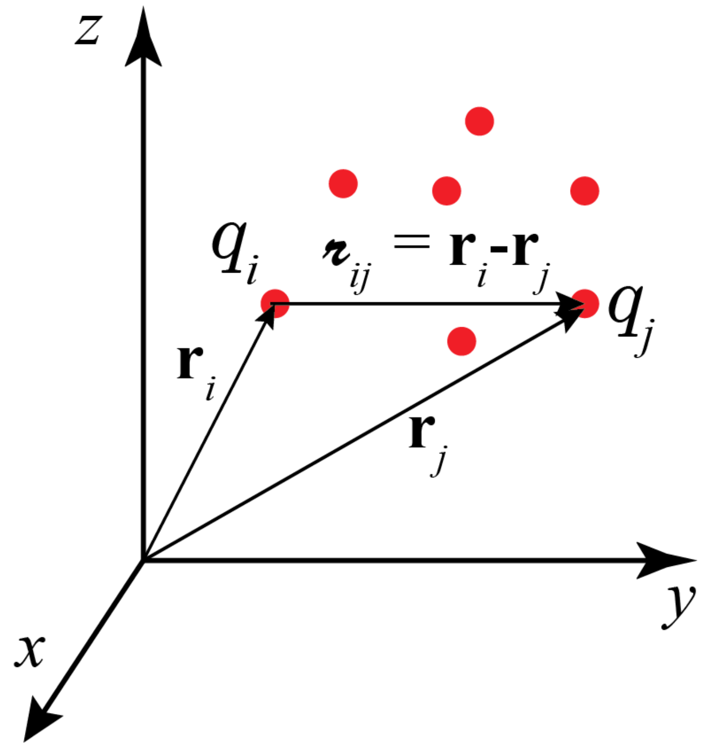
Total work required to bring in the first four charges:

$$W = W_1 + W_2 + W_3 + W_4 = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{q_1 q_3}{\mathscr{r}_{13}} + \frac{q_2 q_3}{\mathscr{r}_{23}} + \frac{q_1 q_4}{\mathscr{r}_{14}} + \frac{q_2 q_4}{\mathscr{r}_{24}} + \frac{q_3 q_4}{\mathscr{r}_{34}}\right)$$

Total work required to bring in n point charges, with charge $q_1, q_2, q_3 \cdots q_n$, respectively:

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j>i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}} \quad = \frac{1}{2}\times\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

$$= \frac{1}{2}\times\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n} q_i\left(\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{q_j}{\mathscr{r}_{ij}}\right)$$



$$\boxed{W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r_i})}$$

- This is the total work required to assemble n point charges
- The potential $V(\mathbf{r_i})$ is the potential at $\mathbf{r_i}$ due to all charges, except the charge at $\mathbf{r_i}$.

The Work required to assemble a continuous charge Distribution:

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i\, V(\mathbf{r_i})$$

- This is the total worked required to assemble n **point** charges
- The potential $V(\mathbf{r_i})$ is the potential at $\mathbf{r_i}$ due to all the other charges, except the charge at $\mathbf{r_i}$.

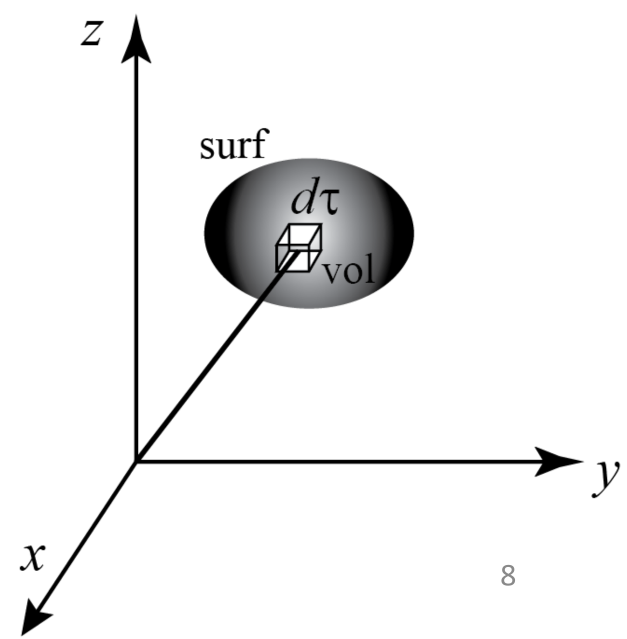
What would be the required work if it is a continuous distribution of charge ?

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i\, V(\mathbf{r_i}) \;\rightarrow\; \frac{1}{2}\int_{vol} dq\, V(\mathbf{r})$$

$$\rightarrow\; \frac{1}{2}\int_{vol} \rho\, V(\mathbf{r})\, d\tau$$

Is this correct? Not really !

The potential $V(\mathbf{r})$ inside the integral is the potential at point $\mathbf{r}$. However, the potential $V(\mathbf{r_i})$ inside the summation in the potential at $\mathbf{r_i}$ due to all the charges except the charge at $\mathbf{r_i}$. Because of this difference in the definition of the potentials, the integral formula turns out to be different.



The Work required to assemble a continuous charge Distribution:

$$W = \frac{1}{2}\int_{vol} \rho\, V d\tau \;= \frac{\epsilon_0}{2}\int_{vol} (\nabla\cdot\mathbf{E})\, V d\tau \qquad \left[\text{Using } \rho = \epsilon_0(\nabla\cdot\mathbf{E})\right]$$

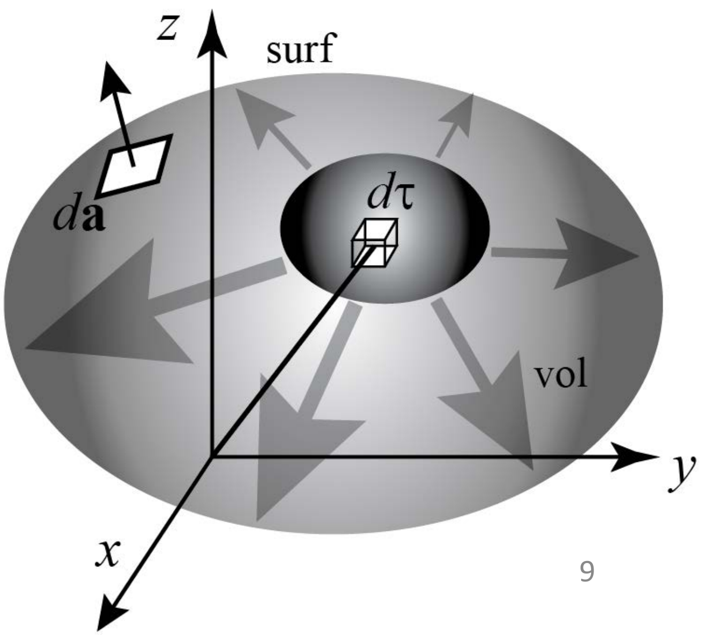
$$W = -\frac{\epsilon_0}{2}\int_{vol} \mathbf{E}\cdot\nabla V d\tau + \frac{\epsilon_0}{2}\int_{vol} \nabla\cdot V\mathbf{E} d\tau \qquad \left(\begin{array}{l}\text{Using the product rule}\\ \nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)\end{array}\right)$$

$$W = -\frac{\epsilon_0}{2}\int_{vol} \mathbf{E}\cdot\nabla V d\tau + \frac{\epsilon_0}{2}\oint_{surf} V\mathbf{E}\cdot d\mathbf{a} \qquad \left(\begin{array}{l}\text{Using the divergence theorem}\\ \int_{Vol}(\nabla\cdot\mathbf{A})d\tau = \oint_{Surf}\mathbf{A}\cdot d\mathbf{a}\end{array}\right)$$

$$W = \frac{\epsilon_0}{2}\int_{vol} E^2 d\tau + \frac{\epsilon_0}{2}\oint_{surf} V\mathbf{E}\cdot d\mathbf{a} \qquad \left[\text{Using } -\nabla V = \mathbf{E}\right]$$

$$\boxed{W = \frac{\epsilon_0}{2}\int_{all\ space} E^2 d\tau}$$

When the volume we are integrating over is very large, the contribution due to the surface integral is negligibly small.



The Energy of a Continuous Charge Distribution:

$$W = \frac{1}{2}\int_{vol} \rho V d\tau \qquad W = \frac{\epsilon_0}{2}\int_{all\ space} E^2 d\tau$$

So, what is the energy of a point charge using the above formula?

Electric field of a point charge is $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$

The energy of a point charge is therefore,

$$W = \frac{\epsilon_0}{2}\int_{all\ space}\left(\frac{1}{4\pi\epsilon_0}\right)^2\left(\frac{q}{r^2}\right)^2 r^2 \sin\theta dr d\theta d\phi = \frac{q^2}{8\pi\epsilon_0}\int_{all\ space}\frac{1}{r^2}dr$$

$= \infty$?? due to incorrect conversion of the sum into an integral

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r_i}) \rightarrow \frac{1}{2}\int_{vol} \rho V d\tau$$

This is the total work done to assemble a set of point charges. This does not include the self energy of assembling a point charge.

This is the total energy of a charge distribution including the self energy of assembling the charge distribution. Assembling a point charge requires infinite energy. This is why this expression gives infinity for the energy of a point charge.

The Electrostatic Energy (Summary):

The work required to construct a system of a point charge Q is: $W = QV(\mathbf{r})$

Total work required to put together n **point** charges is: $$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r_i})$$

$V(\mathbf{r_i})$ is the potential at $\mathbf{r_i}$ due to all charges, except the charge at $\mathbf{r_i}$.

The Energy of a Continuous Charge Distribution: $$\mathrm{W} = \frac{1}{2}\int_{vol} \rho\, V d\tau$$
$$= \frac{\epsilon_0}{2}\int_{all\ space} E^2 d\tau$$

Note # 1: The total work required to assemble a continuous charge distribution = the total energy of a continuous charge distribution

Note # 2: The self energy of assembling a point charge is infinite. Therefore, the total energy of a point charge is infinite.

Note # 3: For systems consisting of point charges, we do not talk about the total energy. We only discuss the total work required to put together the system.