Semester II, 2017-18
Department of Physics, IIT Kanpur

PHY103A: Lecture # 7

(Text Book: Intro to Electrodynamics by Griffiths, 3rd Ed.)

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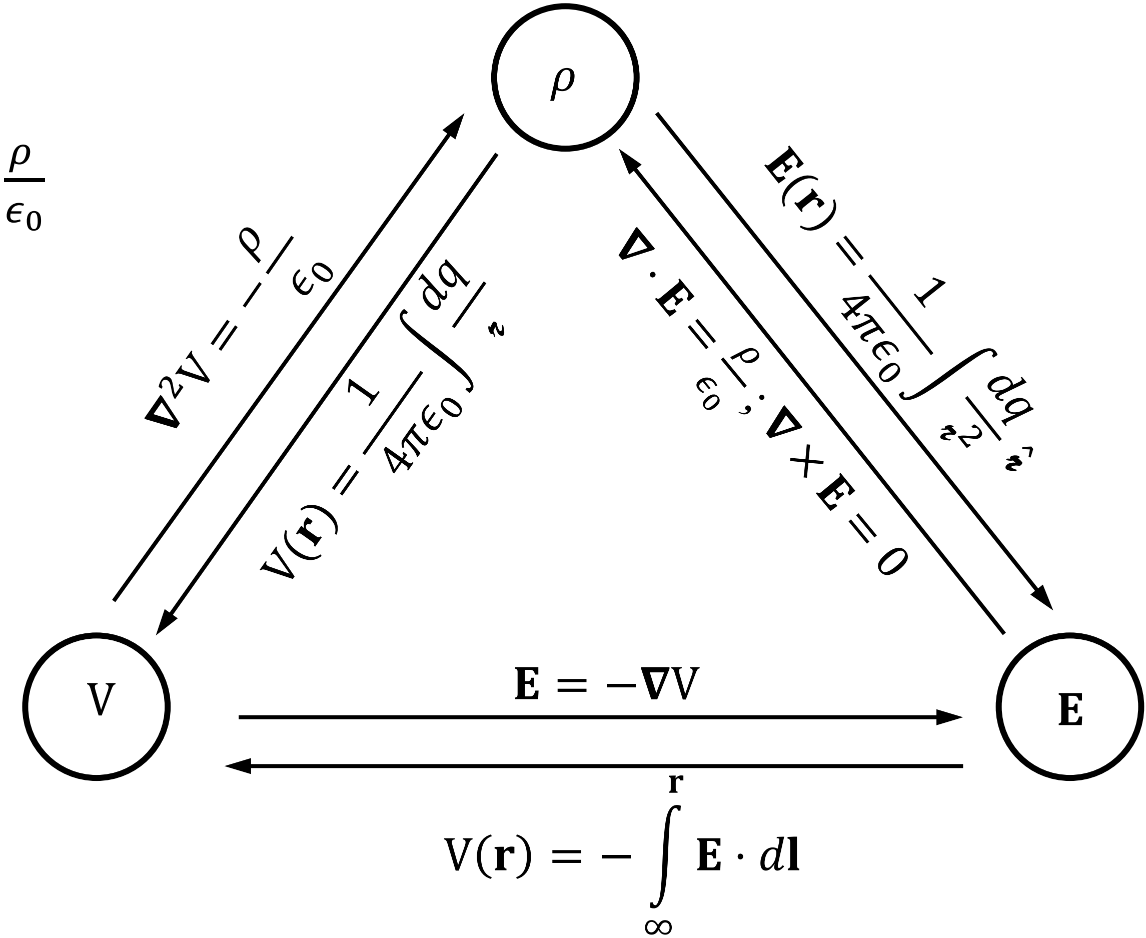
Notes

- HW # 3 is uploaded on the course webpage.
- Solutions to HW#2 have also been uploaded.
- Class this Saturday (Jan 20th).
- Are lecture notes sufficient for exams?
- Office Hours??

Summary of Lecture # 6:

- Poisson's Equation: $\nabla^2 V = -\frac{\rho}{\epsilon_0}$



- The work required to construct a system of one point charge Q:

$$W = QV(\mathbf{r})$$

- Total work required to put together n **point** charges:

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r_i})$$

- Energy of a continuous charge distribution:

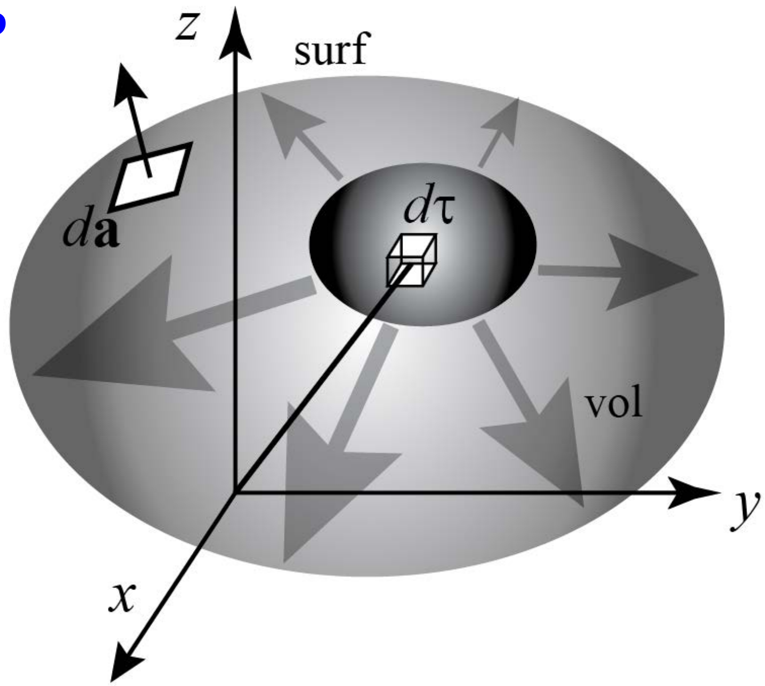
$$W = \frac{1}{2}\int_{vol} \rho V d\tau = \frac{\epsilon_0}{2}\int_{all\ space} E^2 d\tau$$

Where is the electrostatic energy stored ?

The Energy of a Continuous Charge Distribution:

$$W = \frac{1}{2}\int_{vol} \rho\, V d\tau$$

$$= \frac{\epsilon_0}{2}\int_{all\ space} E^2 d\tau$$



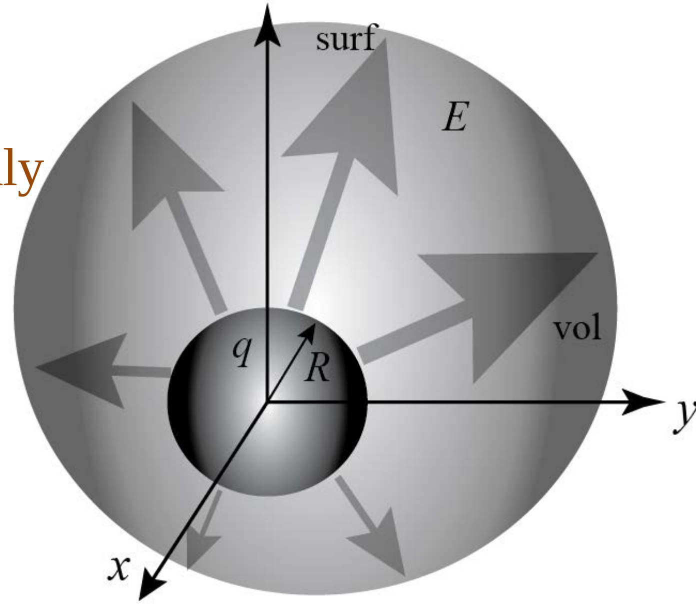
- The first expression has a volume integral of $\rho\, V$ over the localized space whereas the second one has the volume integral of E^2over all space.

So, where is the electrostatic energy stored? Within the localized charge distribution or over all space?

- Just as both the integrals are mathematically correct, both the interpretations are also correct. The electrostatic energy can be interpreted as stored locally within the charge distribution or globally over all space.
- Again, at this point, as regarding fields, we know how to calculate different physical quantities but we don't really know what exactly the field is.

Where is the electrostatic energy stored ?

Ex. 2.8 (Griffiths, 3rd Ed.): Find the energy of a uniformly charged **spherical shell** of total charge q and radius R.



$$W_{\text{shell}} = \frac{1}{2}\int_{vol} \rho\, V d\tau \;= \frac{1}{2}\int \sigma\, V da$$

$$= \frac{1}{2}\int \left(\frac{q}{4\pi\, R^2}\right) \times \left(\frac{q}{4\pi\epsilon_0 R}\right) R^2 \sin\theta\; d\theta\; d\phi$$

$$= \frac{1}{2}\left(\frac{q}{4\pi\, R^2}\right) \times \left(\frac{q}{4\pi\epsilon_0 R}\right) R^2\; 4\pi = \frac{q^2}{8\pi\epsilon_0 R}$$

Alternatively:

$$W_{\text{shell}} = \frac{\epsilon_0}{2}\int_{all\ space} E^2 d\tau$$

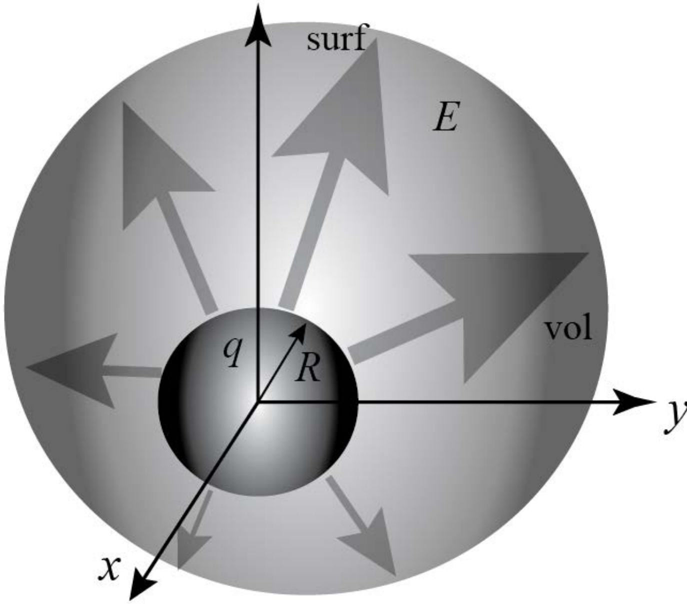
What is E? $\mathbf{E} = 0$ inside and $\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2}\,\hat{\boldsymbol{r}}$ outside

$$W_{\text{shell}} = \frac{\epsilon_0}{2}\int_{all\ space} \left(\frac{q}{4\pi\epsilon_0 r^2}\right)^2 r^2 \sin\theta\; dr d\theta\; d\phi = \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2 4\pi \int_{r=R}^{\infty} \frac{1}{r^2} dr$$

$$= \frac{q^2}{8\pi\epsilon_0}\frac{1}{R}$$

The two expressions for energy indeed give us the same information

Where is the electrostatic energy stored ?



Ex. 2.8 (Griffiths, 3rd Ed.): Find the energy of a uniformly charged **solid sphere** of total charge q and radius R.

$$W_{\text{sphere}} = \frac{\epsilon_0}{2}\int_{all\ space} E^2 d\tau$$

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2}\,\hat{\boldsymbol{r}}, \text{ outside}$$

HW Prob 2.5(a): $\mathbf{E} = \frac{\rho r}{3\epsilon_0}\,\hat{\boldsymbol{r}} = \frac{q}{4\pi R^3/3}\frac{r}{3\epsilon_0}\,\hat{\boldsymbol{r}} = \frac{qr}{4\pi\epsilon_0 R^3}\,\hat{\boldsymbol{r}}$, inside

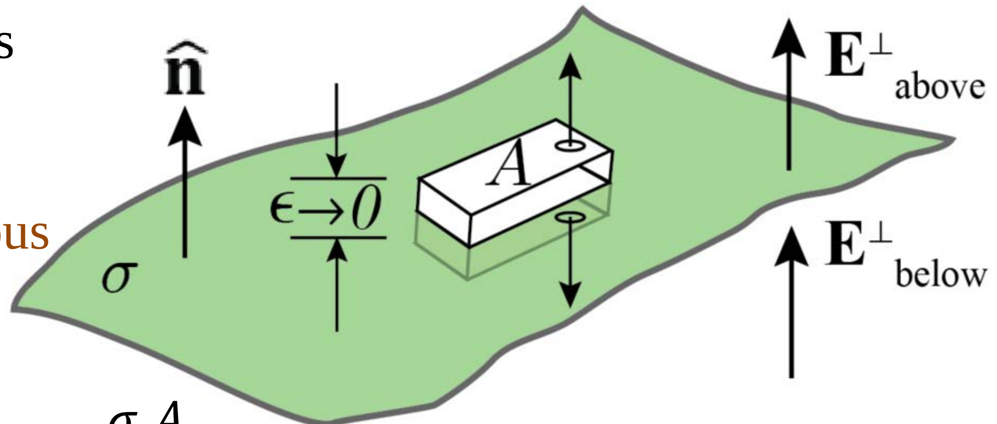
$$W_{\text{sphere}} = \frac{\epsilon_0}{2}\frac{1}{(4\pi\epsilon_0)^2}q^2\left\{\int_0^R \frac{1}{r^4}4\pi r^2 dr + \int_R^\infty \frac{r^2}{R^6}4\pi r^2 dr\right\} = \frac{q^2}{4\pi\epsilon_0}\frac{3}{5R}$$

Recall: $W_{\text{shell}} = \frac{q^2}{8\pi\epsilon_0}\frac{1}{R} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{2R}$

For the same charge and radius, a solid sphere has more total energy than a spherical shell.

Electrostatic Boundary Conditions (Consequences of the fundamental laws):

How does electric field ($\mathbf{E}$) change across a boundary containing surface charge σ?



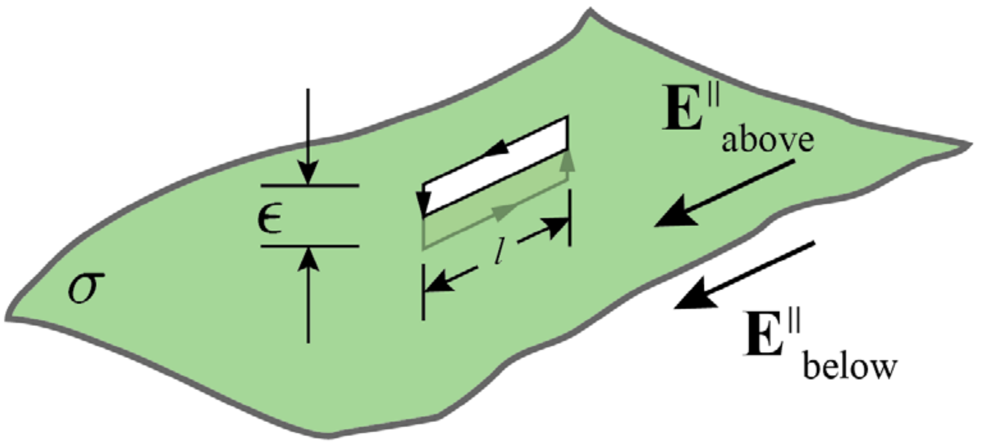
1. Normal component of E is Discontinuous

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \longleftrightarrow \oint_{surf} \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{enc}}{\epsilon_0}$$

$$\mathbf{E}^{\perp}{}_{above} A - \mathbf{E}^{\perp}{}_{below} A + 0 + 0 + 0 + 0 = \frac{\sigma A}{\epsilon_0}$$

$$\boxed{\mathrm{E}^{\perp}{}_{above} - \mathrm{E}^{\perp}{}_{below} = \frac{\sigma}{\epsilon_0}}$$

2. Parallel component of E is Continuous



$$\nabla \times \mathbf{E} = 0 \longleftrightarrow \oint_{path} \mathbf{E} \cdot d\mathbf{l} = \mathbf{0}$$

$$\mathbf{E}^{||}{}_{above}\, l - \mathbf{E}^{||}{}_{below}\, l + 0 + 0 = 0$$

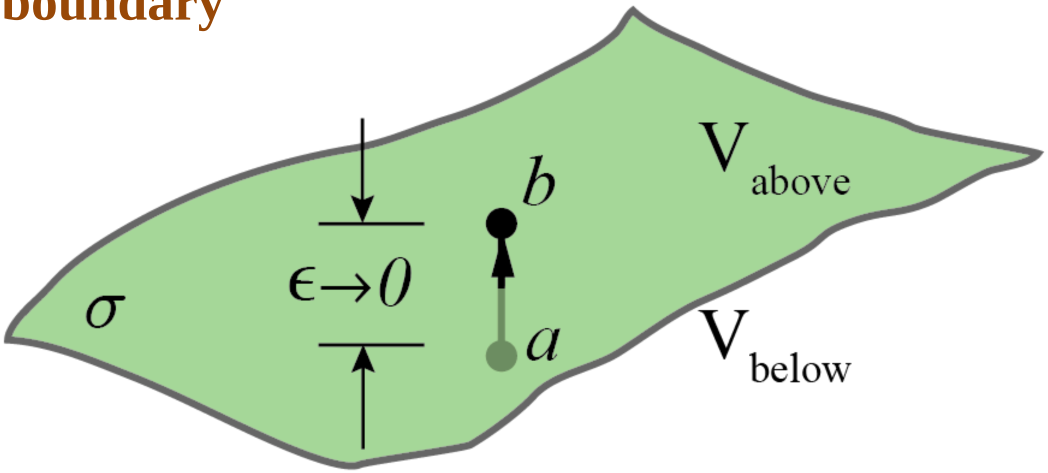
$$\boxed{\mathrm{E}^{||}{}_{above} - \mathrm{E}^{||}{}_{below} = 0}$$

The electrostatic boundary condition $\boxed{\mathbf{E}_{above} - \mathbf{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}}$

Electrostatic Boundary Conditions (Consequences of the fundamental laws):

How does electric potential (V) change across a boundary containing surface charge σ?

3. Potential V is continuous across a boundary



$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\boldsymbol{a}}^{\boldsymbol{b}} \mathbf{E} \cdot d\mathbf{l}$$

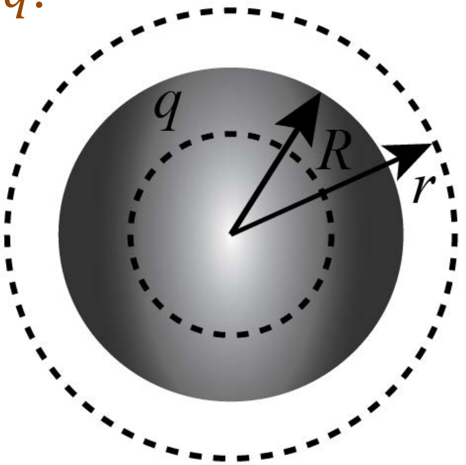
$$V_{above} - V_{below} = -\int_{\boldsymbol{a}}^{\boldsymbol{b}} \mathbf{E} \cdot d\mathbf{l}$$

For $\epsilon \to 0$, $\quad -\int_{\boldsymbol{a}}^{\boldsymbol{b}} \mathbf{E} \cdot d\mathbf{l} = 0$

$$\boxed{V_{above} - V_{below} = 0}$$

Uniformly charged spherical shell

Ex. 2.6 (Griffiths, 3rd Ed.): Find the electric field and electric potential inside and outside a uniformly charged sphere of radius R and total charge q.



The electric field outside the shell: $\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}}$

The electric field inside the shell: $\mathbf{E}(\mathbf{r}) = 0$

The electric potential at a point outside the shell ($r > R$):

$$\mathrm{V}(\mathbf{r}) = -\int_{\infty}^{r} \mathbf{E}\cdot d\mathbf{l} = -\int_{\infty}^{r} \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\,dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

The electric potential for a point inside the shell ($r < R$):

$$\mathrm{V}(\mathbf{r}) = -\int_{\infty}^{R} \mathbf{E}\cdot d\mathbf{l} - \int_{R}^{r} \mathbf{E}\cdot d\mathbf{l} = -\int_{\infty}^{R} \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\,dr' - 0 = \frac{1}{4\pi\epsilon_0}\frac{q}{R}$$

Check the boundary condition on Electric field at $r = R$

$$\mathrm{E}^{\perp}{}_{\text{above}} - \mathrm{E}^{\perp}{}_{\text{below}} = \frac{\sigma}{\epsilon_0} \Rightarrow \mathrm{E}^{\perp}{}_{\text{above}} - 0 = \frac{q}{\epsilon_0\, 4\pi R^2} \Rightarrow \mathrm{E}^{\perp}{}_{\text{above}} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}$$ **OK** ✓

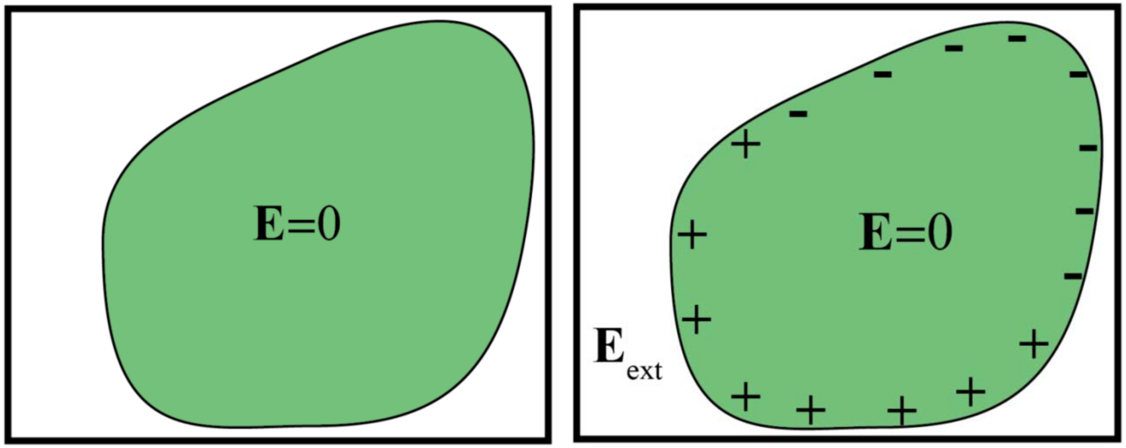
Check the boundary condition on Electric potential

$$\mathrm{V}_{\text{above}} - \mathrm{V}_{\text{below}} = 0 \Rightarrow \frac{1}{4\pi\epsilon_0}\frac{q}{R} - \frac{1}{4\pi\epsilon_0}\frac{q}{R} = 0$$ **OK** ✓

Conductors (Materials containing unlimited supply of electrons)

(1) The electric field $\mathbf{E} = 0$ inside a conductor

This is true even when the conductor is placed in an external electric field $\mathbf{E}_{\mathbf{ext}}$.



(2) The charge density $\rho = 0$ inside a conductor.

This is because $\mathbf{E} = 0$ inside a conductor and therefore $\rho = \epsilon_0 \nabla \cdot \mathbf{E} = 0$.

(3) Any net charge resides on the surface.

Why?
To minimize the energy

$$W_{\text{sphere}} = \frac{q^2}{4\pi\epsilon_0}\frac{3}{5R} > W_{\text{shell}} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{2R}$$

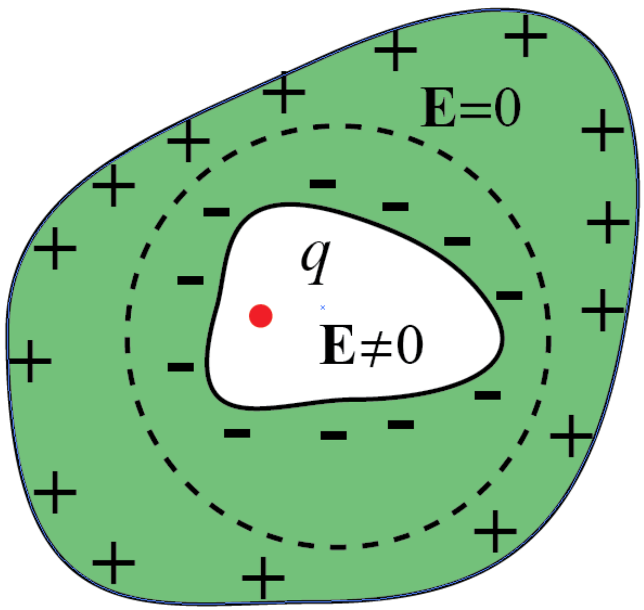
(4) A conductor is an equipotential.

This is because $\mathbf{E} = 0$. So, for any two points $\mathbf{a}$ and $\mathbf{b}$,

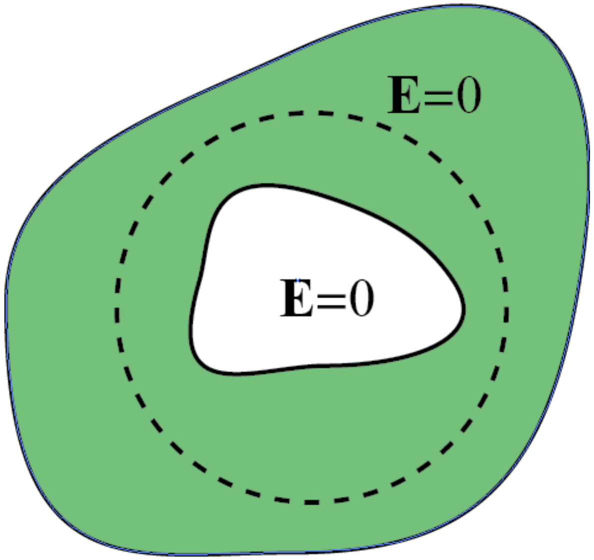
$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\boldsymbol{a}}^{\boldsymbol{b}} \mathbf{E} \cdot d\mathbf{l} = 0.$$ This means $V(\mathbf{b}) = V(\mathbf{a})$.

(5) $\mathbf{E}$ is perpendicular to the surface, just outside the conductor.

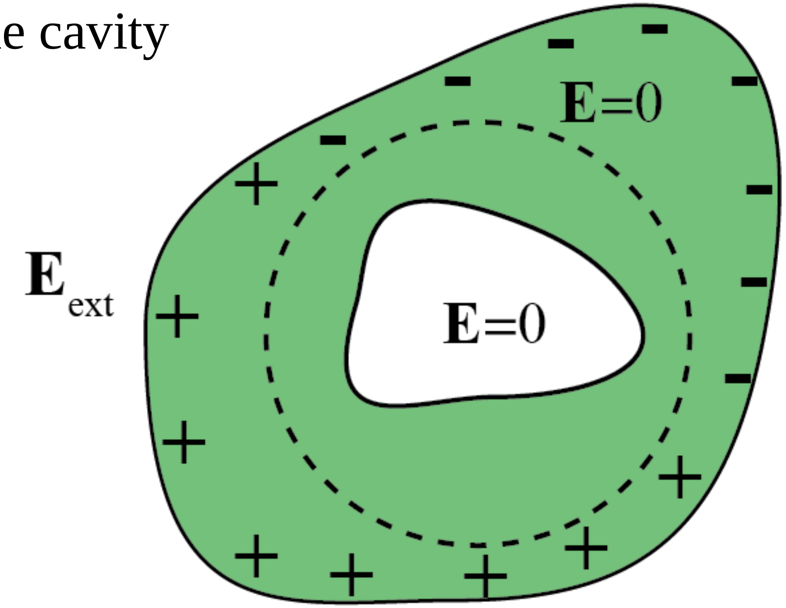
Induced Charges



Charge inside the cavity



No charge inside the cavity



No charge inside the cavity,
Conductor in an external field

Induced Charges

Prob. 2.36 (Griffiths, 3rd Ed.):

- Surface charge σ_a? $\sigma_a = -\dfrac{q_a}{4\pi a^2}$

- Surface charge σ_b? $\sigma_b = -\dfrac{q_b}{4\pi b^2}$

- Surface charge σ_R? $\sigma_R = \dfrac{q_a + q_b}{4\pi R^2}$

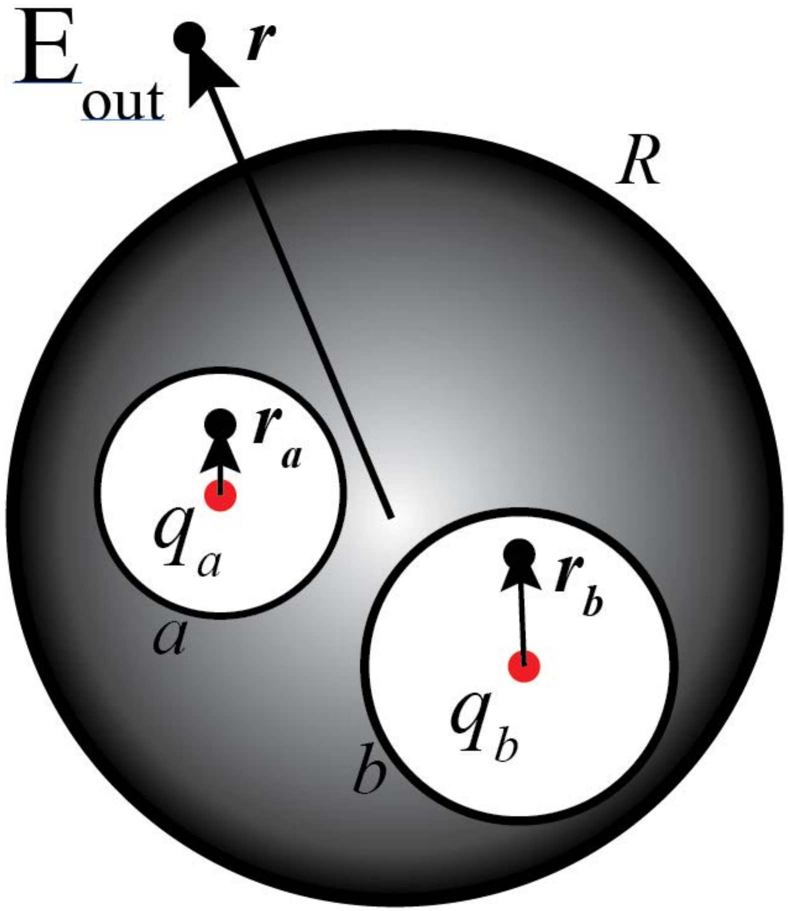
- $\mathbf{E}(\mathbf{r}_a)$? $\mathbf{E}_{\text{out}} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_a}{{r_a}^2}\hat{\mathbf{r}}_a$

- $\mathbf{E}(\mathbf{r}_b)$? $\mathbf{E}_{\text{out}} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_b}{{r_b}^2}\hat{\mathbf{r}}_b$

- $\mathbf{E}_{\text{out}}(\mathbf{r})$? $\mathbf{E}_{\text{out}} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_a + q_b}{r^2}\hat{\mathbf{r}}$

- Force on q_a ? 0

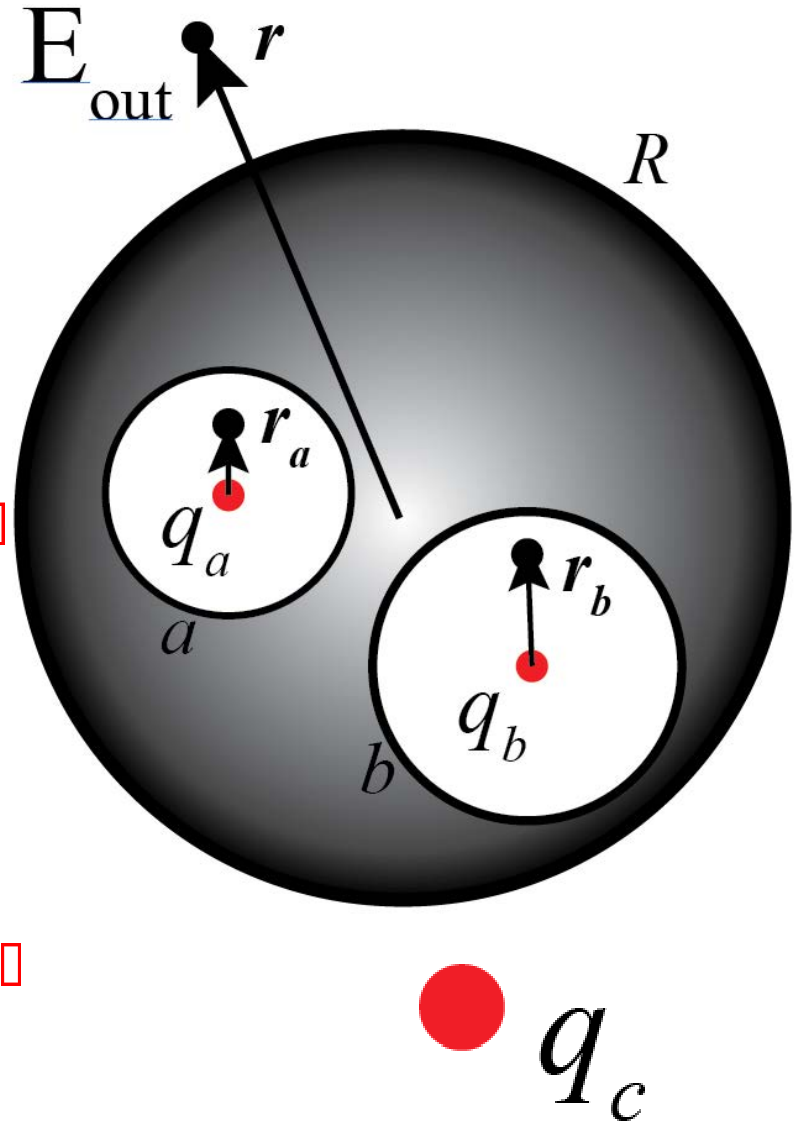
- Force on q_b ? 0



Induced Charges

Prob. 2.36 (Griffiths, 3rd Ed.):

- Surface charge σ_a? $\sigma_a = -\frac{q_a}{4\pi a^2}$ **Same ✓**
- Surface charge σ_b? $\sigma_b = -\frac{q_b}{4\pi b^2}$ **Same ✓**
- Surface charge σ_R? $\sigma_R = \frac{q_a + q_b}{4\pi R^2}$ **Changes □**
- $\mathbf{E}(\mathbf{r}_a)$? $\mathbf{E}_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{q_a}{{r_a}^2}\hat{\mathbf{r}}_a$ **Same ✓**
- $\mathbf{E}(\mathbf{r}_b)$? $\mathbf{E}_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{q_b}{{r_b}^2}\hat{\mathbf{r}}_b$ **Same ✓**
- $\mathbf{E}_{\text{out}}(\mathbf{r})$? $\mathbf{E}_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{q_a + q_b}{r^2}\hat{\mathbf{r}}$ **Changes □**
- Force on q_a ? 0 **Same ✓**
- Force on q_b ? 0 **Same ✓**



Bring in a third charge q_c