Semester II, 2017-18
Department of Physics, IIT Kanpur

PHY103A: Lecture # 8

(Text Book: Intro to Electrodynamics by Griffiths, 3rd Ed.)

Anand Kumar Jha
19-Jan-2018

Summary of Lecture # 7:

- Electrostatic Boundary Conditions

Electric Field:

$$\left.\begin{aligned} E^{\perp}{}_{above} - E^{\perp}{}_{below} &= \frac{\sigma}{\epsilon_0} \\ E^{\|}{}_{above} - E^{\|}{}_{below} &= 0 \end{aligned}\right\} \quad \mathbf{E}_{above} - \mathbf{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

Electric Potential: $V_{above} - V_{below} = 0$

- Basic Properties of Conductors

(1) The electric field $\mathbf{E} = 0$ inside a conductor, always.

(2) The charge density $\rho = 0$ inside a conductor. This is because $\rho = \epsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E} = 0$.

(3) Any net charge resides on the surface. Why? To minimize the energy.

(4) A conductor is an equipotential.

(5) $\mathbf{E}$ is perpendicular to the surface, just outside the conductor.

Summary of Lecture # 7:

Prob. 2.36 (Griffiths, 3rd Ed.):

- Surface charge σ_a? $\sigma_a = -\frac{q_a}{4\pi a^2}$

- Surface charge σ_b? $\sigma_b = -\frac{q_b}{4\pi b^2}$

- Surface charge σ_R? $\sigma_R = \frac{q_a + q_b}{4\pi R^2}$

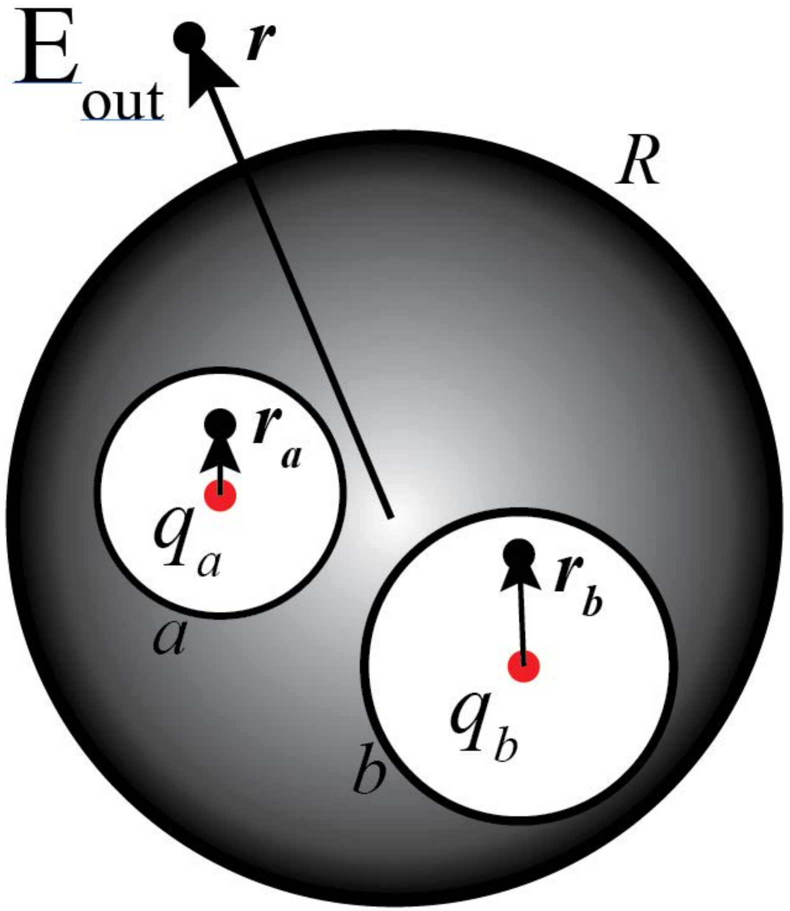
- $\mathbf{E}(\mathbf{r}_a)$? $\mathbf{E}_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{q_a}{{r_a}^2}\hat{\mathbf{r}}_a$

- $\mathbf{E}(\mathbf{r}_b)$? $\mathbf{E}_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{q_b}{{r_b}^2}\hat{\mathbf{r}}_b$

- $\mathbf{E}_{\text{out}}(\mathbf{r})$? $\mathbf{E}_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{q_a + q_b}{r^2}\hat{\mathbf{r}}$

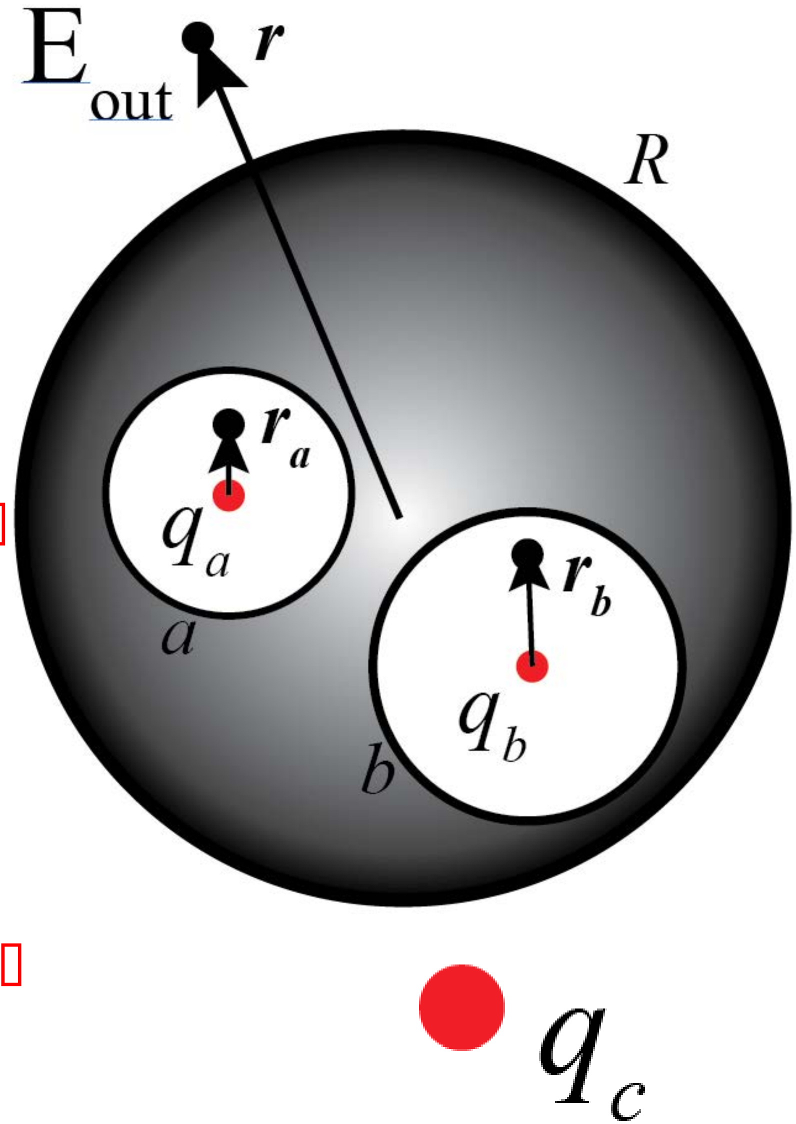
- Force on q_a ? 0

- Force on q_b ? 0



Summary of Lecture # 7:

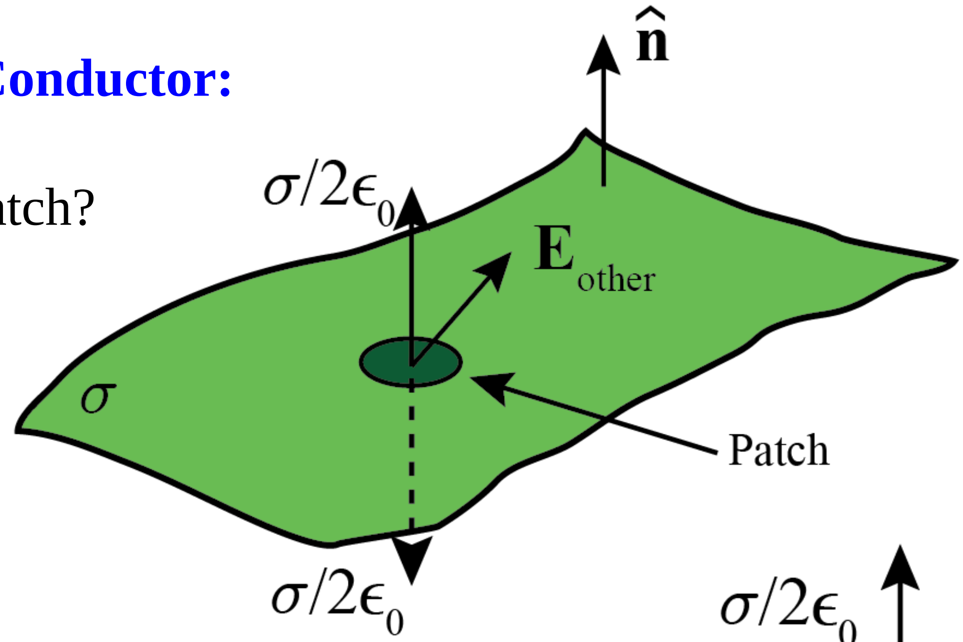
Prob. 2.36 (Griffiths, 3rd Ed.):



- Surface charge σ_a? $\sigma_a = -\dfrac{q_a}{4\pi a^2}$ **Same ✓**
- Surface charge σ_b? $\sigma_b = -\dfrac{q_b}{4\pi b^2}$ **Same ✓**
- Surface charge σ_R? $\sigma_R = \dfrac{q_a + q_b}{4\pi R^2}$ **Changes □**
- $\mathbf{E}(\mathbf{r}_a)$? $\mathbf{E}_{\text{out}} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_a}{{r_a}^2}\hat{\mathbf{r}}_a$ **Same ✓**
- $\mathbf{E}(\mathbf{r}_b)$? $\mathbf{E}_{\text{out}} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_b}{{r_b}^2}\hat{\mathbf{r}}_b$ **Same ✓**
- $\mathbf{E}_{\text{out}}(\mathbf{r})$? $\mathbf{E}_{\text{out}} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_a + q_b}{r^2}\hat{\mathbf{r}}$ **Changes □**
- Force on q_a ? 0 **Same ✓**
- Force on q_b ? 0 **Same ✓**

Bring in a third charge q_c

Surface Charge and the Force on a Conductor:



What is the electrostatic force on the patch?

Force per unit area on the patch is:

$$\mathbf{F} = \sigma\, \mathbf{E}\,(?) = \sigma\, \mathbf{E}_{\text{other}}$$

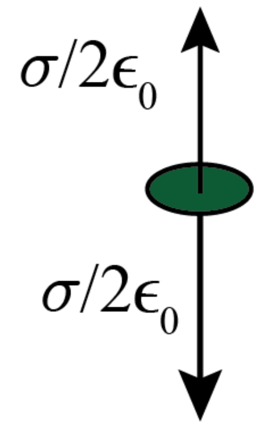
$$\mathbf{E}_{\text{above}} = \mathbf{E}_{\text{other}} + \mathbf{E}_{\text{patch, above}}$$

$$= \mathbf{E}_{\text{other}} + \frac{\sigma}{2\epsilon_0}\,\hat{\mathbf{n}}$$

$$\mathbf{E}_{\text{below}} = \mathbf{E}_{\text{other}} + \mathbf{E}_{\text{patch, below}}$$

$$= \mathbf{E}_{\text{other}} - \frac{\sigma}{2\epsilon_0}\,\hat{\mathbf{n}}$$

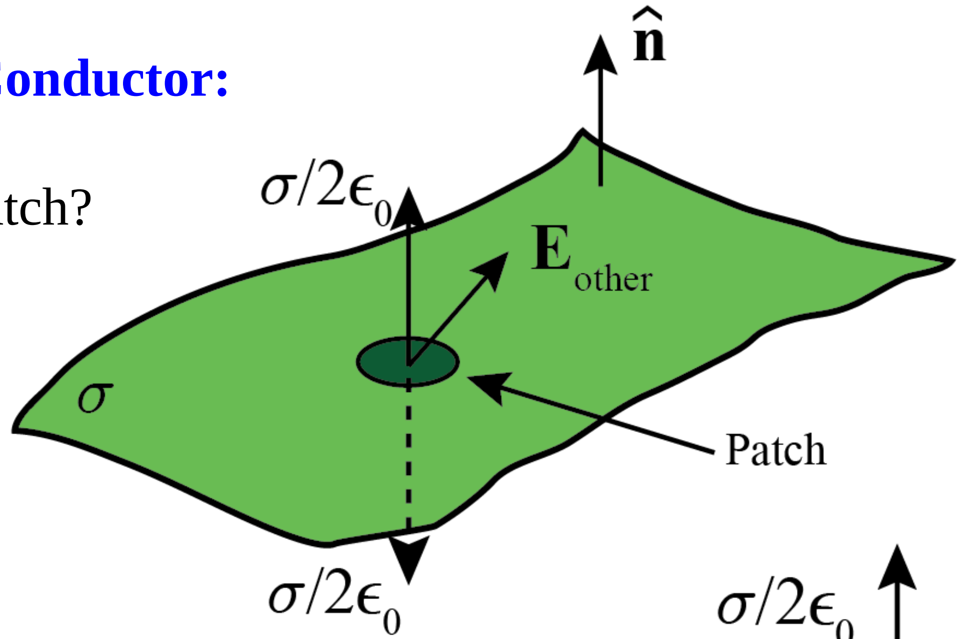
$$\mathbf{E}_{\text{other}} = \frac{1}{2}\left(\mathbf{E}_{\text{above}} + \mathbf{E}_{\text{below}}\right)$$



But, inside a metal, $\mathbf{E} = \mathbf{0}$, so $\mathbf{E}_{\text{below}} = 0$

$$\mathbf{E}_{\text{other}} = \frac{\sigma}{2\epsilon_0}\,\hat{\mathbf{n}} \qquad \mathbf{E}_{\text{above}} = \mathbf{E}_{\text{other}} + \frac{\sigma}{2\epsilon_0}\,\hat{\mathbf{n}} = \frac{\sigma}{\epsilon_0}\,\hat{\mathbf{n}}$$

Surface Charge and the Force on a Conductor:

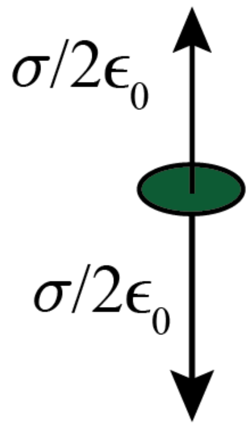


What is the electrostatic force on the patch?

Force per unit area on the patch is:

$$\mathbf{F} = \sigma\,\mathbf{E}\,(?) = \sigma\,\mathbf{E}_{\text{other}}$$

$$\mathbf{E}_{\text{other}} = \frac{\sigma}{2\epsilon_0}\,\hat{\mathbf{n}} \qquad \mathbf{E}_{\text{above}} = \frac{\sigma}{\epsilon_0}\,\hat{\mathbf{n}}$$



Force per unit area on the patch is:

$$\mathbf{F} = \sigma\,\mathbf{E}_{\text{other}} = \frac{\sigma^2}{2\epsilon_0}\,\hat{\mathbf{n}}$$

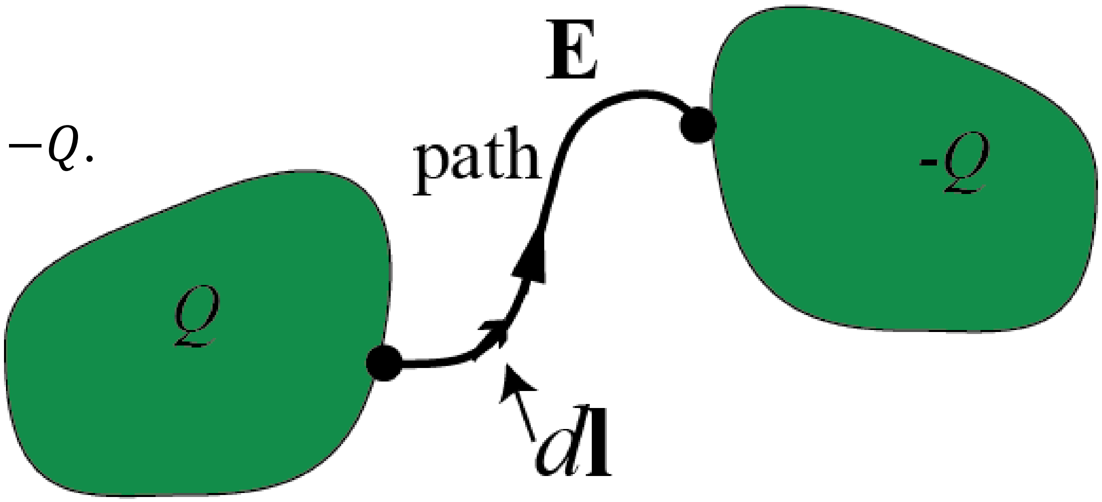
Force per unit area is pressure.

So, the electrostatic pressure is:

$$P = \frac{\sigma^2}{2\epsilon_0} = \frac{\epsilon_0}{2}E^2$$

Capacitor:

Two conductors with charge Q and $-Q$.



What is the potential difference between them?

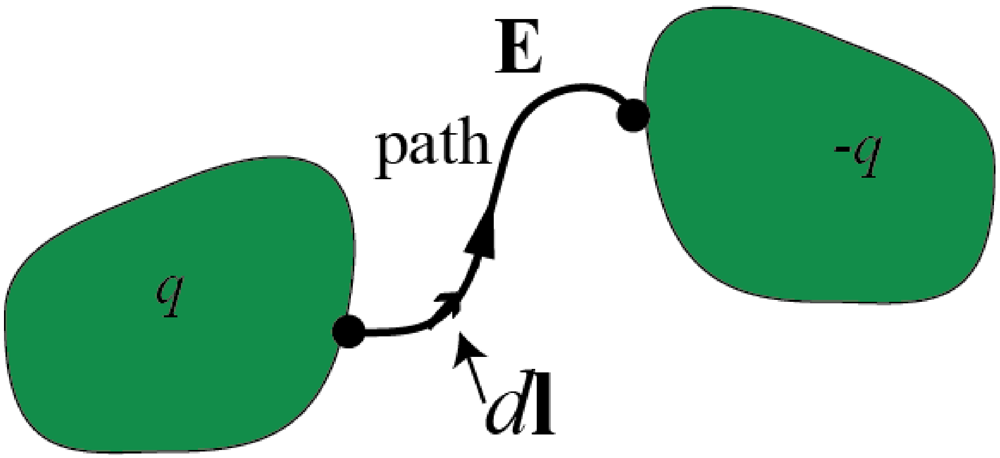
$$V = V_+ - V_- = -\int_{(-)}^{(+)} \mathbf{E}.d\mathbf{l}$$

Capacitance C is defined as: $C \equiv \frac{Q}{V}$

- Capacitance is the ability of a system to store electric charge.
- It is purely a geometric quantity.
- C is measured in farads (F), Coulomb/Volt.
- Practical units are microfarad (10^{-6}) or picofarad (10^{-12}).

Work needed to charge a Capacitor:

Two conductors with charge q and $-q$.



How much work needs to be done to increase the charge by dq

Recall | The work required to create a system of a point charge Q: $W = QV(\mathbf{r})$

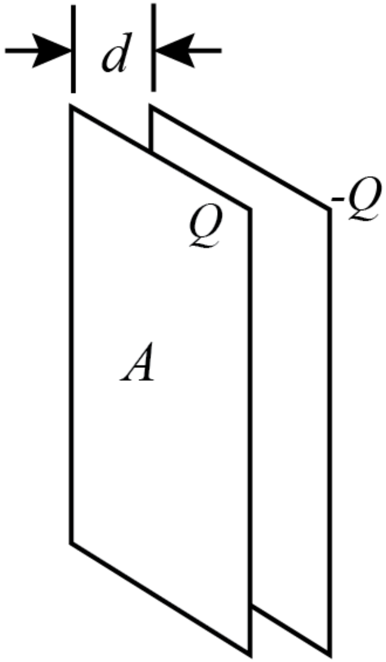
$$dW = V dq = \left(\frac{q}{C}\right) dq$$

The work necessary to go from $q = 0$ to $q = Q$ is

$$W = \int_0^Q \left(\frac{q}{C}\right) dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} C V^2$$

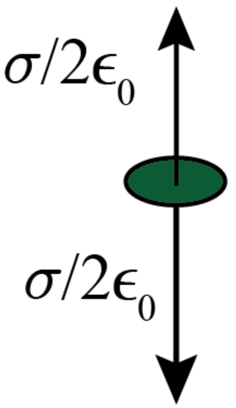
Capacitor:

Ex. 2.10 (Griffiths, 3rd Ed.): Find the capacitance of a parallel plate capacitor. Area = A, Separation = d



The electric field between the plates

$$E = \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0} = \frac{Q}{A\epsilon_0}$$



The potential difference is therefore,

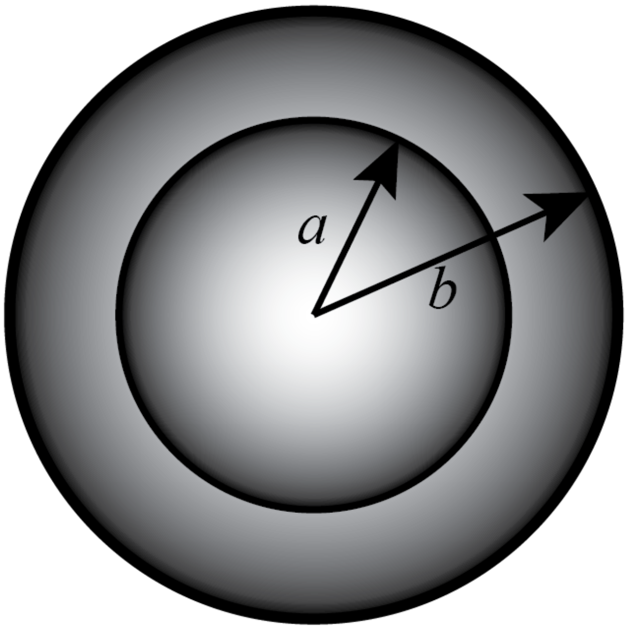
$$V = -\int \mathbf{E}.d\mathbf{l} = E\ d = \frac{Q}{A\epsilon_0} d$$

Capacitance C is:

$$C = \frac{Q}{V} = \frac{A\epsilon_0}{d}$$

Capacitor:

Ex. 2.11 (Griffiths, 3rd Ed.): Find the capacitance of two concentric spherical metal shells, with radii a and b.

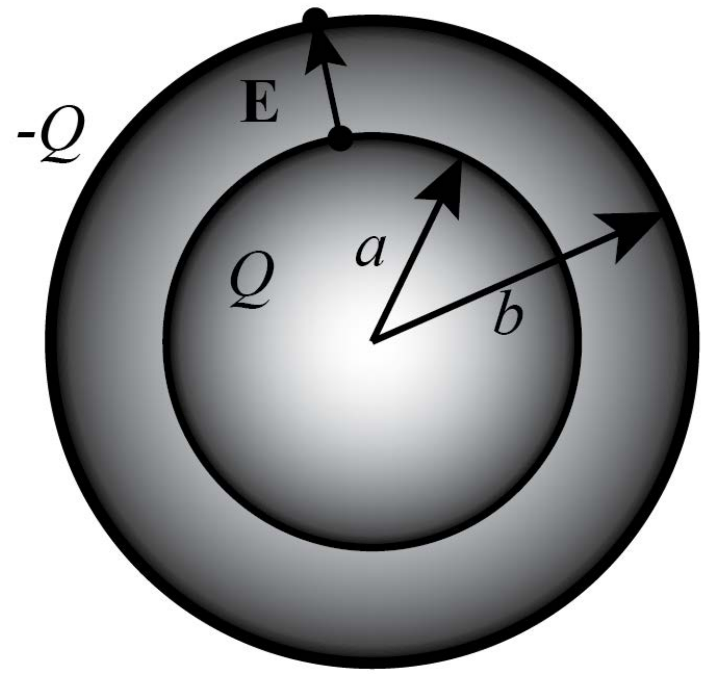


Suppose there is charge Q on the inner shell and $-Q$ on the outer shell.

The electric field between the two shells is

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{\boldsymbol{r}}$$

The potential difference is therefore,



$$V = V_b - V_a = -\int_a^b \mathbf{E}.\,d\mathbf{l} = -\int_a^b \frac{Q}{4\pi\epsilon_0 r^2}\,dr$$

$$= -\frac{Q}{4\pi\epsilon_0}\int_a^b \frac{1}{r^2}\,dr \quad = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Capacitance is: $C = \dfrac{Q}{V} \; = 4\pi\epsilon_0 \dfrac{ab}{(b-a)}$

Superposition principle for electrostatic energy:

We have seen several electrostatic systems, including conductors.

We know how to calculate electrostatic energy for different system.

We know that electric field ($\mathbf{E}$) and electric potential (V) follow the principle of superposition.

$$\mathbf{E} = \mathbf{E_1} + \mathbf{E_2} + \cdots \qquad V = \mathrm{V}_1 + \mathrm{V}_2 + \cdots$$

Does electrostatic energy also follow the principle of superposition? No

Why? Because W is quadratic in E?

$$W = \frac{\epsilon_0}{2}\int E^2\,d\tau = \frac{\epsilon_0}{2}\int (\mathbf{E_1} + \mathbf{E_2})^2\,d\tau = \frac{\epsilon_0}{2}\int (E_1^2 + E_2^2 + 2\mathbf{E_1}\cdot\mathbf{E_2})\,d\tau$$

$$= \frac{\epsilon_0}{2}\int E_1^2\,d\tau + \frac{\epsilon_0}{2}\int E_2^2\,d\tau + \epsilon_0\int \mathbf{E_1}\cdot\mathbf{E_2}\,d\tau$$

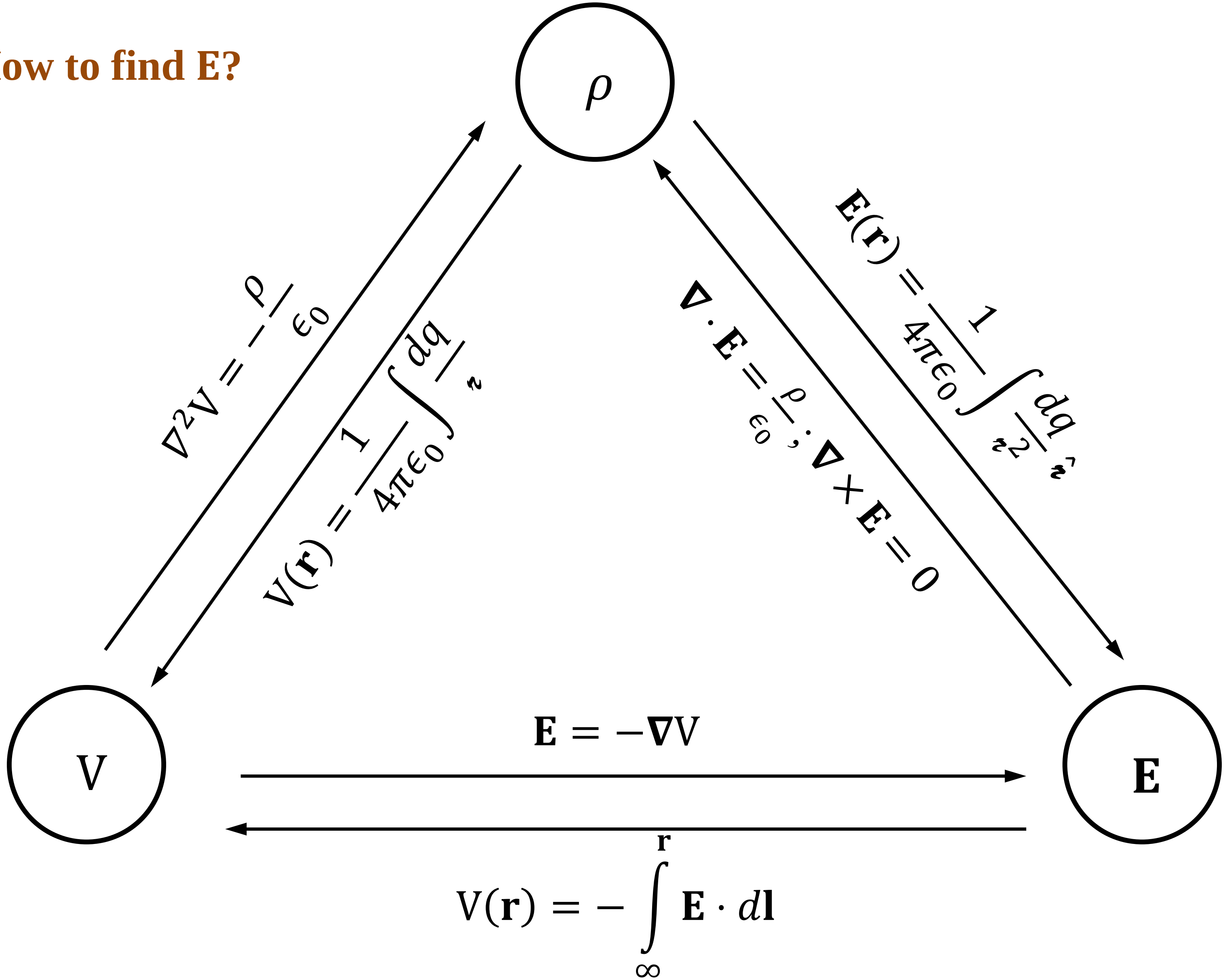
$$= \quad W_1 \quad + W_2 \quad + \epsilon_0\int \mathbf{E_1}\cdot\mathbf{E_2}\,d\tau$$

Electric Field: Can We See It?

https://www.youtube.com/watch?v=7vnmL853784

Special Techniques (Electrostatics):

How to find E?



Special Techniques (Electrostatics):

- Laplace's Equation

- The Method of Images

- Multipole Expansion

Laplace's Equation

Q: How to find electric field **E** ?

Ans: $$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}^2}\hat{\mathscr{r}}$$ (Coulomb's Law)

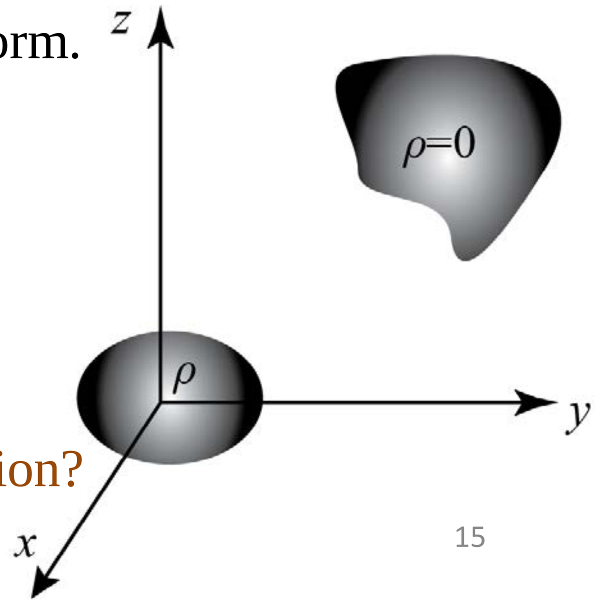
Very difficult to calculate the integral except for very simple situation

Alternative: First calculate the electric potential

$$\mathrm{V}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}}$$

This integral is relatively easier but in general still difficult to handle

Alternative: Express the above equation in the different form.



$$\nabla^2 \mathrm{V} = -\frac{\rho}{\epsilon_0}$$ (Poisson's Equation)

When $\rho = 0$ $\quad \nabla^2 \mathrm{V} = 0$ (Laplace's Equation)

If $\rho = 0$ everywhere, $V = 0$ everywhere

If ρ is localized, what is V away from the charge distribution?

Laplace's Equation in One Dimension

$$\nabla^2 V = 0 \qquad \text{(Laplace's Equation)}$$

In Cartesian coordinates,

$$\frac{\partial^2}{\partial x^2}V + \frac{\partial^2}{\partial y^2}V + \frac{\partial^2}{\partial z^2}V = 0$$

If $V(x, y, z)$ depends on only one variable, x, We have

$$\frac{d^2}{dx^2}V = 0$$

(One-dimensional Laplace's Equation, ordinary differential equation)

General Solution: $V(x) = mx + b$

How to calculate the constants m and b ?

Using boundary conditions

What decides the boundary condition?

The charge distribution