Semester II, 2017-18
Department of Physics, IIT Kanpur

# PHY103A: Lecture # 9

(Text Book: Intro to Electrodynamics by Griffiths, 3rd Ed.)

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## Summary of Lecture # 8:

- Force per unit area on a conductor: $\mathbf{F} = \sigma\, \mathbf{E}_{\text{other}} = \dfrac{\sigma^2}{2\epsilon_0}\, \hat{\mathbf{n}}$

- The electrostatic pressure on a conductor: $P = \dfrac{\sigma^2}{2\epsilon_0} = \dfrac{\epsilon_0}{2} E^2$

- Capacitance $C$ is defined as: $C \equiv \dfrac{Q}{V}$

- The work necessary to charge a capacitor upto charge $Q$: $W = \dfrac{1}{2}\dfrac{Q^2}{C} = \dfrac{1}{2} C V^2$

- The electrostatic interaction energy: $W = W_1 + W_2 + \epsilon_0 \int \mathbf{E_1} \cdot \mathbf{E_2}\, d\tau$

- Applications of Electrostatics:
  (i) Faraday Cage
  (ii) Capacitor

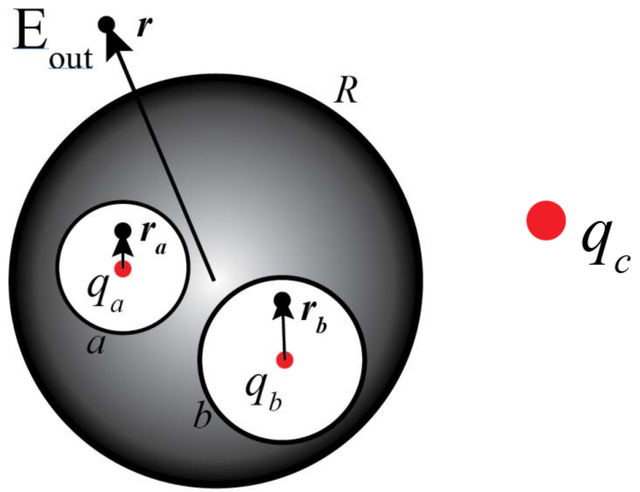
- Special techniques: Laplace's Equation in one-dimension

## Questions 1:

Does the force on $q_a$ depend on $q_c$, if the cavity was not spherical

Ans: No

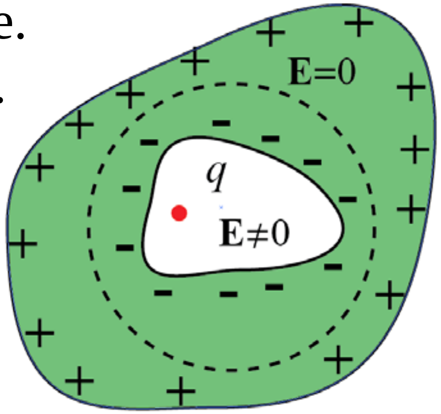
- Force on $q_a$ ? 0
- Force on $q_b$ ? 0




## Questions 2:

If $q = e$, how does induced charge distribute itself on the inner surface?

Ans: Within classical electrodynamics, the induced charge will distribute as usual.
In quantum electrodynamics, $e$ is the minimum charge.
Induced charge density is interpreted probabilistically.



# Special Techniques: Laplace's Equation

Q: How to find electric field **E** ?

Ans: $$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}^2}\hat{\mathscr{r}}$$ (Coulomb's Law)

Very difficult to calculate the integral except for very simple situation

Alternative: First calculate the electric potential

$$\mathrm{V}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}}$$

This integral is relatively easier but in general still difficult to handle

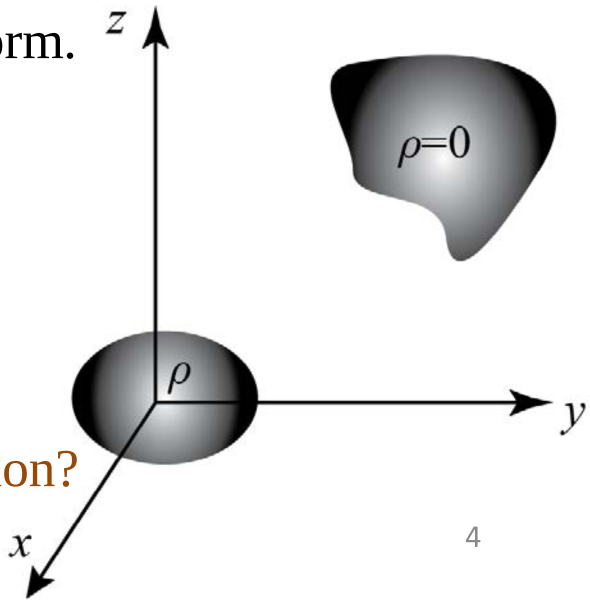
Alternative: Express the above equation in the different form.

$$\nabla^2 \mathrm{V} = -\frac{\rho}{\epsilon_0}$$ (Poisson's Equation)

When $\rho = 0$ $$\nabla^2 \mathrm{V} = 0$$ (Laplace's Equation)

If $\rho = 0$ everywhere, $V = 0$ everywhere

If $\rho$ is localized, what is $V$ away from the charge distribution?



# Laplace's Equation

$$\nabla^2 V = 0$$

(Laplace's Equation)

$$\frac{\partial^2}{\partial x^2}V + \frac{\partial^2}{\partial y^2}V + \frac{\partial^2}{\partial z^2}V = 0$$

(In Cartesian coordinates)

If the potential V($\mathbf{r}$) is a solution to the Laplace's equation then V($\mathbf{r}$) is the average value of potential over a spherical surface of radius $R$ centered at $\mathbf{r}$.

$$V(\mathbf{r}) = \frac{1}{4\pi R^2}\oint_{sphere} V da$$

As a result, V($\mathbf{r}$) cannot have local maxima or minima; the extreme values of V($\mathbf{r}$) must occur at the boundaries.

Why? Because if the potential has a maximum value $V_{max}(\mathbf{r})$ at $\mathbf{r}$ then one could draw a small sphere around $\mathbf{r}$ such that every value of potential on that sphere and thus the average would be smaller than $V_{max}(\mathbf{r})$

## Laplace's Equation

$$\nabla^2 V = 0$$

(Laplace's Equation)

$$\frac{\partial^2}{\partial x^2}V + \frac{\partial^2}{\partial y^2}V + \frac{\partial^2}{\partial z^2}V = 0$$

(In Cartesian coordinates)

If the potential V(**r**) is a solution to the Laplace's equation then V(**r**) is the average value of potential over a spherical surface of radius $R$ centered at **r**.

**Proof:**

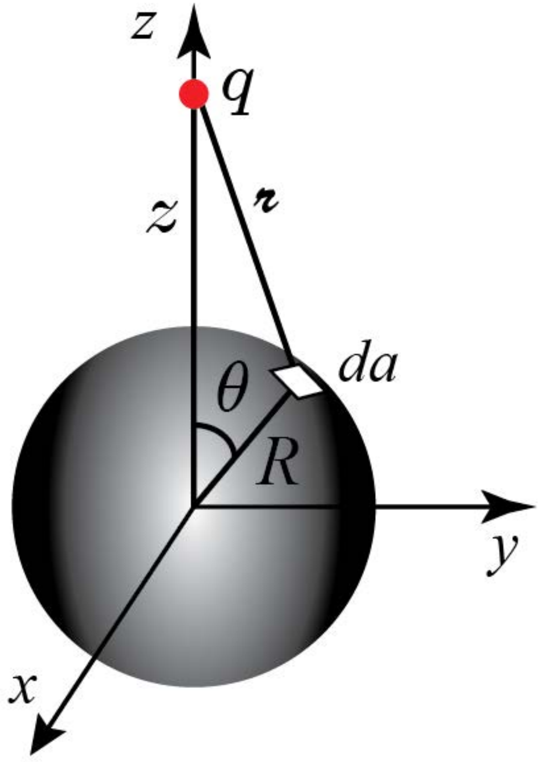
$$V(\mathbf{r}) = \frac{1}{4\pi R^2}\oint_{sphere} V da$$

$$= \frac{1}{4\pi R^2}\int_0^{\pi}\int_0^{2\pi} \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}} R^2 \sin\theta d\theta d\phi$$

$$= \frac{2\pi}{4\pi R}\frac{q}{4\pi\epsilon_0}\int_0^{\pi}\frac{R}{\sqrt{z^2+R^2-2zR\cos\theta}}\sin\theta d\theta$$

$$= \frac{2\pi}{4\pi R}\frac{q}{4\pi\epsilon_0}\int_0^{\pi}\frac{d}{d\theta}\left(\frac{1}{z}\sqrt{z^2+R^2-2zR\cos\theta}\right) d\theta$$

$$= \frac{2\pi}{4\pi R}\frac{q}{4\pi\epsilon_0 z}[(z+R)-(z-R)] = \frac{q}{4\pi\epsilon_0 z}$$



## Laplace's Equation

$$\nabla^2 V = 0$$

(Laplace's Equation)

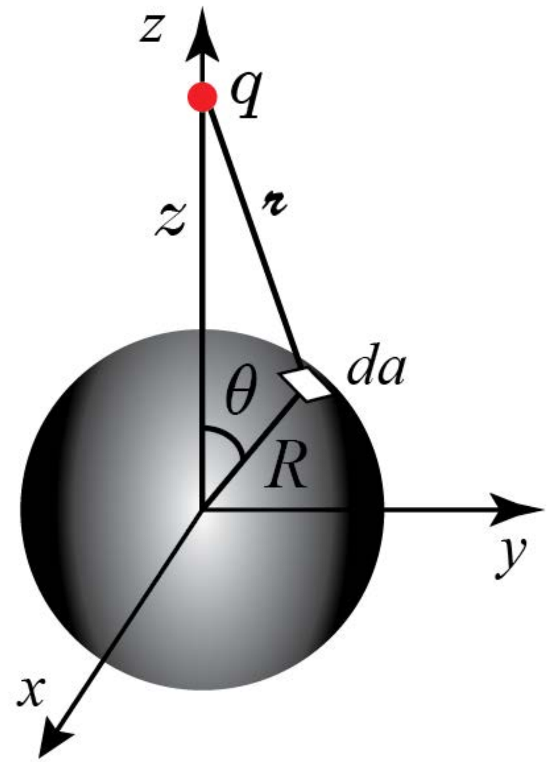
$$\frac{\partial^2}{\partial x^2}V + \frac{\partial^2}{\partial y^2}V + \frac{\partial^2}{\partial z^2}V = 0$$

(In Cartesian coordinates)

If the potential V(**r**) is a solution to the Laplace's equation then V(**r**) is the average value of potential over a spherical surface of radius *R* centered at **r**.

**Proof:**

$$V(\mathbf{r}) = \frac{1}{4\pi R^2}\oint_{sphere} V da \quad = \frac{q}{4\pi\epsilon_0 z}$$




We have proved the theorem for a point charge but since potentials follow the principle of linear superposition, the theorem is proved for any arbitrary charge distribution.

**QED**

## Laplace's Equation in two dimensions

If the potential $V(x, y)$ is a solution to the Laplace's equation then $V(x, y)$ is the average value of potential over a circle of radius $R$ centered at $(x, y)$.

$$V(x, y) = \frac{1}{2\pi R} \oint_{circle} V dl$$

As a result, $V(x, y)$ cannot have local maxima or minima; the extreme values of $V(x, y)$ must occur at the boundaries.

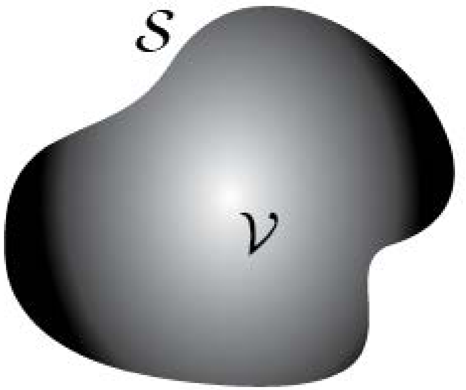
## Laplace's Equation in one dimension

If the potential $V(x)$ is a solution to the Laplace's equation then $V(x)$ is the average of the potential at $x + a$ and $x - a$

$$V(x, y) = \frac{1}{2}[V(x + a) + V(x - a)]$$

As a result, $V(x)$ cannot have local maxima or minima; the extreme values of $V(x)$ must occur at the end points.

# Laplace's Equation (Solutions without solving it)




First Uniqueness Theorem: The solution to Laplace's Equation in some volume $\mathcal{V}$ is uniquely determined if V is specified on the boundary surface $\mathcal{S}$.

Proof: Suppose $V_1$ and $V_2$ are two distinct solutions to Laplace's equation within volume $\mathcal{V}$ with the same value on the boundary surface $\mathcal{S}$.

$\nabla^2 V_1 = 0$ and $\nabla^2 V_2 = 0$

$V_3 \equiv V_1 - V_2$ $\quad \nabla^2 V_3 = \nabla^2 (V_1 - V_2) = \nabla^2 V_1 - \nabla^2 V_2 = 0$

$V_3$ also satisfies Laplaces's equation.

What is the value of $V_3$ at the boundary surface $\mathcal{S}$ ?

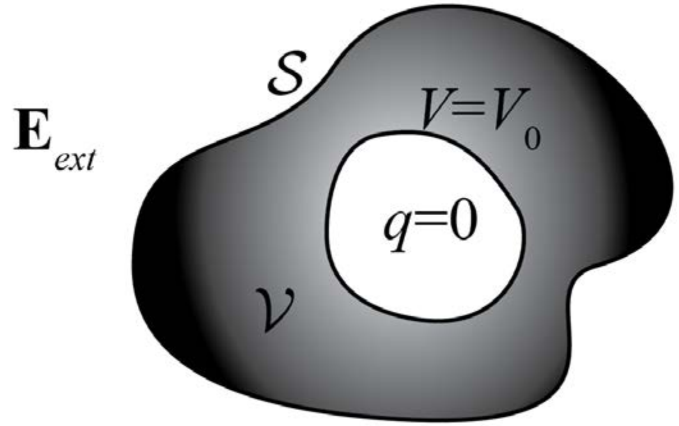
0 (Because at the boundary, $V_1 = V_2$. Hence, $V_3 = V_1 - V_2 = 0$)

But Laplace's equation does not allow for any local extrema.

So, since $V_3 = 0$ at the boundary, $V_3$ must be 0 everywhere.

Hence $V_1 = V_2$ **QED**

Ex. 3.1 (Griffiths, 3rd Ed. ): What is the potential inside an enclosure with no charge and surrounded completely by a conducting material?



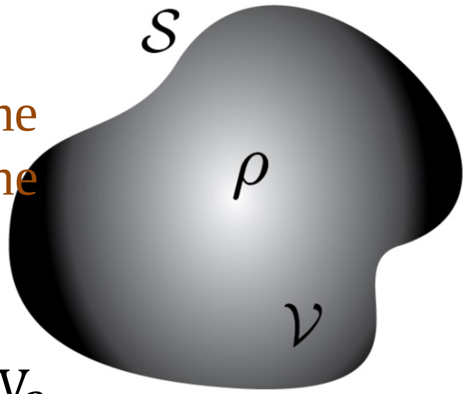

Potential on the cavity-wall is a constant $\mathrm{V} = \mathrm{V}_0$.

Note 1: $\mathrm{V} = \mathrm{V}_0$ is ***a*** solution of the Laplaces’s equation inside the cavity

Note 2: $\mathrm{V} = \mathrm{V}_0$ satisfies the conditions on the boundary surface.

Therefore, $\mathrm{V} = \mathrm{V}_0$ must be ***the*** solution of the problem.

# Laplace's Equation (Solutions without solving it)




Corollary to First Uniqueness Theorem: The potential in a volume is uniquely determined if (a) the charge desity throughout the region and (b) the value of V at all boundaries, are specified.

Proof: Suppose $V_1$ and $V_2$ are two distinct solutions to Poisson's equation in a region with volume $\mathcal{V}$ and charge density $\rho$. $V_1$ and $V_2$ have the same value at the boundary surface $\mathcal{S}$.

$$\nabla^2 V_1 = -\frac{\rho}{\epsilon_0} \quad \text{and} \quad \nabla^2 V_2 = -\frac{\rho}{\epsilon_0}$$

$$V_3 \equiv V_1 - V_2 \qquad \nabla^2 V_3 = \nabla^2 (V_1 - V_2) = \nabla^2 V_1 - \nabla^2 V_2 = 0$$

$V_3$ satisfies Laplaces's equation.

What is the value of $V_3$ at the boundary surface $\mathcal{S}$ ?

0 (Because at the boundary, $V_1 = V_2$. Hence, $V_3 = V_1 - V_2$=0)

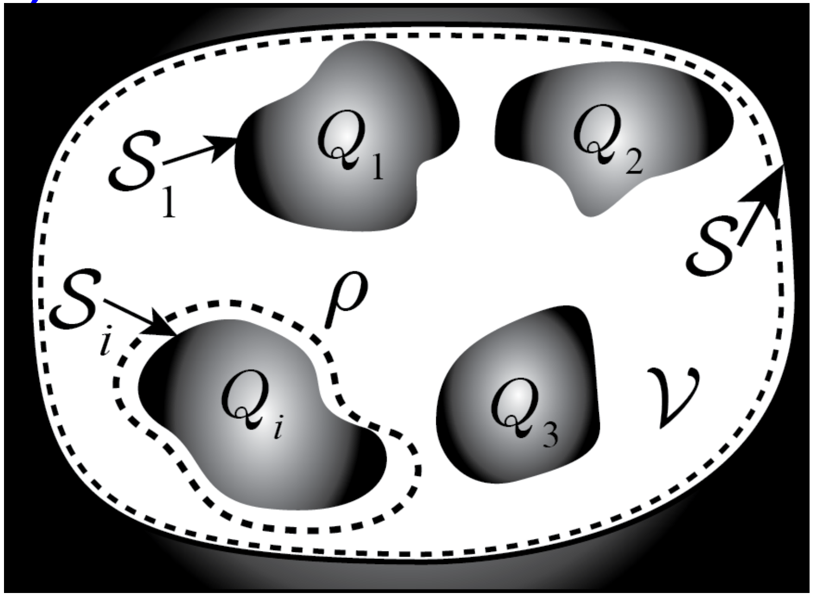
But Laplace's equation does not allow for any local extrema.

So, if $V_3 = 0$ at the boundary, it must be 0 everywhere.

Hence $V_1 = V_2$ **QED**

## Laplace's Equation (Solutions without solving it)

Second Uniqueness Theorem: In a volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density $\rho$, the electric field is uniquely determined if the total charge on each conductor is given.




**Proof:** Suppose $\mathbf{E_1}$ and $\mathbf{E_2}$ are two distinct electric fields satisfying the above conditions.

$$\nabla \cdot \mathbf{E_1} = \frac{\rho}{\epsilon_0} \quad \text{and} \quad \nabla \cdot \mathbf{E_2} = \frac{\rho}{\epsilon_0}$$ (Gauss's law in space between the conductors)

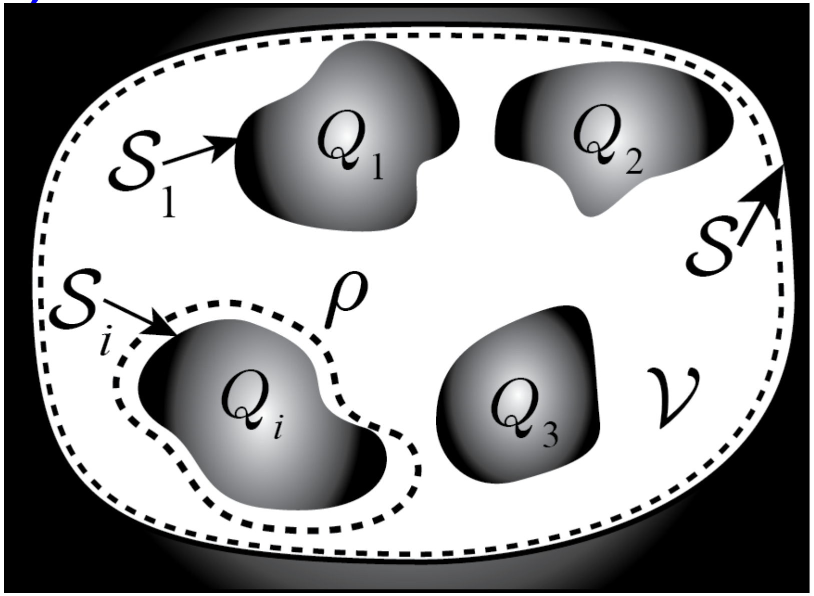
$$\oint_{\mathcal{S}_i} \mathbf{E_1} \cdot d\mathbf{a} = \frac{Q_i}{\epsilon_0} \quad \text{and} \quad \oint_{\mathcal{S}_i} \mathbf{E_2} \cdot d\mathbf{a} = \frac{Q_i}{\epsilon_0}$$ (for Gaussian surface enclosing $i^{\text{th}}$ conductor)

$$\oint_{\mathcal{S}} \mathbf{E_1} \cdot d\mathbf{a} = \frac{Q_{total}}{\epsilon_0} \quad \text{and} \quad \oint_{\mathcal{S}} \mathbf{E_2} \cdot d\mathbf{a} = \frac{Q_{total}}{\epsilon_0}$$ (for Gaussian surface enclosing *all* conductors)

$$\mathbf{E_3} \equiv \mathbf{E_1} - \mathbf{E_2} \qquad \nabla \cdot \mathbf{E_3} = \nabla \cdot (\mathbf{E_1} - \mathbf{E_2}) = 0 \qquad \oint_{\mathcal{S}_i} \mathbf{E_3} \cdot d\mathbf{a} = 0$$

# Laplace's Equation (Solutions without solving it)

Second Uniqueness Theorem: In a volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density $\rho$, the electric field is uniquely determined if the total charge on each conductor is given.




**Proof:**

We know, $V_3$ as well as $V_1$ and $V_2$ are all constants over a conducting surface

$$\nabla \cdot (V_3 \mathbf{E_3}) = V_3(\nabla \cdot \mathbf{E_3}) + \mathbf{E_3} \cdot (\nabla V_3) = -(E_3)^2$$ ( Using $\nabla \cdot \mathbf{E_3} = 0$ & $\mathbf{E_3} = -\nabla V_3$)

$$\int_{\mathcal{V}} \nabla \cdot (V_3 \mathbf{E_3}) d\tau = -\int_{\mathcal{V}} (E_3)^2 \, d\tau$$ (Integrating over the entire volume)

$$\oint_{\mathcal{S}} V_3 \mathbf{E_3} \cdot d\mathbf{a} = -\int_{\mathcal{V}} (E_3)^2 \, d\tau$$ (Applying divergence theorem)

$$V_3 \oint_{\mathcal{S}} \mathbf{E_3} \cdot d\mathbf{a} = -\int_{\mathcal{V}} (E_3)^2 \, d\tau$$ (Since $V_3$ is a constant on the outer boundary $\mathcal{S}$)

$$0 = -\int_{\mathcal{V}} (E_3)^2 \, d\tau$$ (Since $\oint_{\mathcal{S}_i} \mathbf{E_3} \cdot d\mathbf{a} = 0$) □ $E_3 = 0$ (everywhere)

□ $\mathbf{E_1} = \mathbf{E_2}$ QED

## Laplace's Equation (Solutions without solving it)

Comments on Uniqueness theorem:

- If certain conditions are fulfilled, the Uniqueness theorems guarantee that the solution is unique.

- Even if ***a*** solution is obtained by mere guess or intuition that satisfies all the necessary conditions, it must be ***the*** unique solution.

- Uniqueness theorems do not directly help solve Laplace's or Poisson's equation. They help establish that the solution is unique.