Department of Physics
IIT Kanpur, Semester II, 2017-18

PHY103A: Physics II | # Solutions # 1 | **Instructors: AKJ & SC**

Solution 1.1:

(a) At the top of the hill, the gradient of the height function should be zero, that is, $\nabla h(x,y) = 0$. This gives

$$\frac{\partial}{\partial x}h(x,y)\hat{\mathbf{x}} + \frac{\partial}{\partial y}h(x,y)\hat{\mathbf{y}} = \frac{h(x,y)}{60}\Big[(2y-6x-18)\hat{\mathbf{x}} + (2x-8y+28)\hat{\mathbf{y}}\Big] = 0,$$

Or,

$$2y - 6x - 18 = 0 \quad \text{and} \quad 2x - 8y + 28 = 0.$$

Solving these two equations we get $x = -2$ and $y = 3$. Thus the top of the hill is located at $x = -2$ and $y = 3$.

(b) The height of a hill is defined as the height of its top, which for the given hill is located at $x = -2$ and $y = 3$. Therefore the height of the hill is $h(-2,3) = e^{11/12} \simeq 2.5$.

(c) The steepest slope at any point is in the direction of the gradient $\nabla h(x,y)$. Now, the gradient vector at $(1,1)$ is $\nabla h(x,y)\Big|_{1,1} = h(1,1)(22/60)(-\hat{\mathbf{x}}+\hat{\mathbf{y}})$. The direction of this gradient vector is given by $\tan\theta = -1$, or $\theta = 135\,^{\circ}$. Therefore the slope is steepest in the direction $\theta = 135\,^{\circ}$.

(d) The slope of $h(x,y)$ at (1,1) in the direction $\mathbf{n}$ is $\nabla h(1,1)\cdot\mathbf{n} = h(1,1)(22/60)(-\hat{\mathbf{x}}+\hat{\mathbf{y}})\cdot(\hat{\mathbf{x}}+\hat{\mathbf{y}}) = 0$.

Solution 1.2:

We have the magnitude of the separation vector given as

$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}.$$

Therefore the gradient is

$$\begin{aligned}\nabla R &= \Big[\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}\Big]\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \\ &= \frac{(x-x')\hat{\mathbf{x}} + (y-y')\hat{\mathbf{y}} + (z-z')\hat{\mathbf{z}}}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} = \frac{\mathbf{R}}{R}.\end{aligned}$$

Thus we see that the gradient is just a unit vector parallel to $\mathbf{R}$.

Solution 1.3:

We have $\nabla\phi = (2xy + z^3)\hat{\mathbf{x}} + x^2\hat{\mathbf{y}} + 3xz^2\hat{\mathbf{z}}$. Using the definition of gradient in the cartesian coordinates system and equating the two sides of the equation, we get

$$\begin{aligned}\frac{\partial\phi(x,y,z)}{\partial x} &= 2xy + z^3, \\ \frac{\partial\phi(x,y,z)}{\partial y} &= x^2, \\ \frac{\partial\phi(x,y,z)}{\partial z} &= 3xz^2.\end{aligned}$$

Integrating the above equations yeilds

$$\phi(x,y,z) = x^2y + xz^3 + f(y,z)$$
$$\phi(x,y,z) = x^2y + g(x,z)$$
$$\phi(x,y,z) = xz^3 + h(x,y),$$

where $f(y,z)$, $g(x,z)$ and $h(x,y)$ are arbitrary functions. Choose, $f(y,z) = 0$, $g(x,z) = xz^3$ and $h(x,y) = x^2y$. The scaler function is

$$\phi(x,y,z) = x^2y + xz^3 + \text{constant}.$$

We note that the scalar function in this case can only be obtained up to a constant. Also, by taking the gradient of this scalar function, we can verify our result.

Solution 1.4:

$$\nabla \ln|\mathbf{r}| = \left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}\right)\ln[\sqrt{x^2+y^2+z^2}] = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{x^2+y^2+z^2} = \frac{\mathbf{r}}{r^2}.$$

$$\nabla \frac{1}{|\mathbf{r}|} = \left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}\right)\frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{-x\hat{\mathbf{x}} - y\hat{\mathbf{y}} - z\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{3/2}} = -\frac{\mathbf{r}}{r^3}. \quad (1)$$

Solution 1.5:

(a)

$$\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^n}\right) = \nabla\cdot\left(\frac{x\hat{x}+y\hat{y}+z\hat{z}}{(x^2+y^2+z^2)^{1/2}}\frac{1}{(x^2+y^2+z^2)^{n/2}}\right)$$
$$= \frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{\frac{n+1}{2}}}\right] + \frac{\partial}{\partial y}\left[\frac{y}{(x^2+y^2+z^2)^{\frac{n+1}{2}}}\right] + \frac{\partial}{\partial z}\left[\frac{z}{(x^2+y^2+z^2)^{\frac{n+1}{2}}}\right]$$
$$= \frac{1-x^2(n+1)(x^2+y^2+z^2)^{-1}}{(x^2+y^2+z^2)^{\frac{n+1}{2}}} + \frac{1-y^2(n+1)(x^2+y^2+z^2)^{-1}}{(x^2+y^2+z^2)^{\frac{n+1}{2}}} + \frac{1-z^2(n+1)(x^2+y^2+z^2)^{-1}}{(x^2+y^2+z^2)^{\frac{n+1}{2}}}$$
$$= \frac{3-(x^2+y^2+z^2)(n+1)(x^2+y^2+z^2)^{-1}}{(x^2+y^2+z^2)^{\frac{n+1}{2}}}$$
$$= \frac{3-n+1}{(x^2+y^2+z^2)^{\frac{n+1}{2}}} = \frac{2-n}{r^{n+1}}.$$

(b) For $n \neq 2$, $\nabla\cdot(\hat{\mathbf{r}}/r^n)$ decreases with r and tends to zero as r approaches infinity. However, for $n = 2$, $\nabla\cdot(\hat{\mathbf{r}}/r^2) = 0$ everywhere except at $\mathbf{r} = 0$. As we see in the above equation, the divergence at $\mathbf{r} = 0$ is of type $0/0$ and is therefore not defined. So, we cannot use the above result for calculating divergence at $\mathbf{r} = 0$. The divergence $\nabla\cdot(\hat{\mathbf{r}}/r^2)$ at $r = 0$ can be calculated by use of Gauss's theorem and in fact can be shown to be infinity. Mathematically, the divergence in this case can be represented as the Dirac-delta function: $\nabla\cdot(\hat{\mathbf{r}}/r^2) = 4\pi\delta^3(\mathbf{r})$.

(c) One example of the vector-fields of type $\mathbf{E} = \hat{\mathbf{r}}/r^2$ is the electric field due to a point charge. Therefore, this type of vector fields are very relevant in electrodynamics.

Solution 1.6: We have $\boldsymbol{\nabla} \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$. Taking the curl on both sides, we get

$$\begin{aligned} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) &= -\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{B}) \\ \text{or,} \quad \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= -\frac{\partial^2 \mathbf{E}}{\partial t^2} \\ \text{or,} \quad \nabla^2 \mathbf{E} &= \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad \text{since} \quad \nabla \cdot \mathbf{E} = 0. \end{aligned}$$

Similarly, one can show that $\nabla^2 \mathbf{B} = \dfrac{\partial^2 \mathbf{B}}{\partial t^2}$ since $\nabla \cdot \mathbf{B} = 0$.

The above two equations are known as the wave-equations and they describe the propagation of electromagnetic fields in vacuum.

Solution 1.7:

We have $\mathbf{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3zx)\hat{\mathbf{z}}$. Stokes theorem says

$$\int (\boldsymbol{\nabla} \times \mathbf{v}) \cdot d\mathbf{a} = \oint \mathbf{v} \cdot d\mathbf{l}.$$

For this problem the area integral needs to be calculated over the shaded region and the line integral needs to be calculated along the closed path consisting of three line segments, as indicated in Fig. 1.

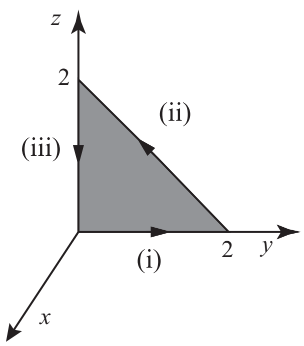

FIG. 1:

Let us first calculate the left hand side of the above equation. We have

$$\boldsymbol{\nabla} \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}} = (0 - 2y)\hat{\mathbf{x}} + (0 - 3z)\hat{\mathbf{y}} + (0 - x)\hat{\mathbf{z}}$$

Therefore,

$$\begin{aligned} \int (\boldsymbol{\nabla} \times \mathbf{v}) \cdot d\mathbf{a} &= \int_{y=0}^{2} \int_{z=0}^{2-y} (-2y\hat{\mathbf{x}} - 3z\hat{\mathbf{y}} - x\hat{\mathbf{z}}) \cdot dydz\hat{\mathbf{x}} \\ &= \int_{y=0}^{2} \int_{z=0}^{2-y} -2ydydz = \int_{y=0}^{2} -2y(2-y)dy = \left[\frac{2y^3}{3} - 2y^2\right]_0^2 = -\frac{8}{3} \end{aligned} \tag{2}$$

We now calculate the line integral along the three line segments.

$$\text{(i)} \quad x = z = 0; \quad \mathbf{v}\cdot d\mathbf{l} = 0; \quad \int \mathbf{v}\cdot d\mathbf{l} = 0$$

$$\text{(ii)} \quad x = 0, z = 2 - y; \quad \mathbf{v}\cdot d\mathbf{l} = 2yzdy; \quad \int \mathbf{v}\cdot d\mathbf{l} = \int_2^0 2yzdy = \int \mathbf{v}\cdot d\mathbf{l} = \int_2^0 2y(2-y)dy = -\frac{8}{3}$$

$$\text{(iii)} \quad x = y = 0; \quad \mathbf{v}\cdot d\mathbf{l} = 0; \quad \int \mathbf{v}\cdot d\mathbf{l} = 0$$

Therefore, we have

$$\oint \mathbf{v}\cdot d\mathbf{l} = 0 - \frac{8}{3} + 0 = -\frac{8}{3} \tag{3}$$

Comparing equations (2) and (3), we verify the Stoke's theorem.

Solution 1.8: We need to verify the divergence theorem, which says that

$$\int (\boldsymbol{\nabla}\cdot\mathbf{v})d\tau = \oint \mathbf{v}\cdot d\mathbf{a}.$$

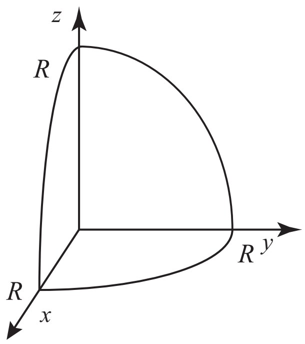

FIG. 2:

For the given problem, the volume integral needs to be performed over the octant of the sphere and the area integral needs to be calculated over the four surfaces that completely surround the octant. Let us first calculate the volume integral. We have $\mathbf{v} = r^2\cos\theta\hat{\mathbf{r}} + r^2\cos\phi\hat{\theta} - r^2\cos\theta\sin\phi\hat{\phi}$. We use the formula for divergence in the spherical coordinate system:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(v_\phi) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r^2\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta r^2\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-r^2\cos\theta\sin\phi) \\
&= \frac{1}{r^2}(4r^3\cos\theta) + \frac{1}{r\sin\theta}(\cos\theta r^2\cos\phi) + \frac{1}{r\sin\theta}(-r^2\cos\theta\cos\phi) \\
&= 4r\cos\theta
\end{aligned}$$

Substituting the above expression for divergence in the volume integral, we get

$$\int (\boldsymbol{\nabla}\cdot\mathbf{v})d\tau = \int (4r\cos\theta)r^2\sin\theta drd\theta d\phi$$

We need to integrate the volume integral over the octant of the sphere as shown in Fig 2. We therefore get,

$$\begin{aligned}\int (\boldsymbol{\nabla}\cdot\mathbf{v})d\tau &= 4\int_0^R r^3 dr \int_0^{\pi/2} \cos\theta\sin\theta d\theta \int_0^{\pi/2} d\phi \\ &= 4\frac{R^4}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{\pi R^4}{4} \end{aligned} \tag{4}$$

The area surrounding the octant consists of four surfaces and thus we need to calculate the area integrals separately over these four surfaces:

(i) The surface in the $x-y$ plane: $d\mathbf{a} = r\sin\theta dr d\phi\hat{\theta}$; $\mathbf{v}\cdot d\mathbf{a} = (r^2\cos\phi)r\sin\theta dr d\phi$. But $\theta = \pi/2$. Therefore, we have

$$\int \mathbf{v}\cdot d\mathbf{a} = \int r^3\cos\phi dr d\phi = \int_0^R r^3 dr \int_0^{\pi/2} \cos\phi d\phi = \frac{R^4}{4}$$

(ii) The surface in the $y-z$ plane: $d\mathbf{a} = r dr d\theta\hat{\phi}$; $\mathbf{v}\cdot d\mathbf{a} = (-r^2\cos\theta\sin\phi) r dr d\theta$. But $\phi = \pi/2$. Therefore, we have

$$\int \mathbf{v}\cdot d\mathbf{a} = -\int r^3\cos\theta dr d\theta = -\int_0^R r^3 dr \int_0^{\pi/2} \cos\theta d\theta = -\frac{R^4}{4}\times 1 = -\frac{R^4}{4}$$

(iii) The surface in the $x-z$ plane: $d\mathbf{a} = -r dr d\theta\hat{\phi}$; $\mathbf{v}\cdot d\mathbf{a} = (-r^2\cos\theta\sin\phi)(-r dr d\theta)$. But $\phi = 0$. Therefore, we have

$$\int \mathbf{v}\cdot d\mathbf{a} = \int 0\cdot d\mathbf{a} = 0$$

(iv) The spherical curved surface: $d\mathbf{a} = r^2\sin\theta d\theta d\phi\hat{\mathbf{r}}$; $\mathbf{v}\cdot d\mathbf{a} = (r^2\cos\theta)(r^2\sin\theta d\theta d\phi)$. But $r = R$. Therefore, we have

$$\int \mathbf{v}\cdot d\mathbf{a} = \int (R^2\cos\theta)(R^2\sin\theta d\theta d\phi) = R^4\int_0^{\pi/2}\cos\theta\sin\theta d\theta \int_0^{\pi/2} d\phi = R^4\times\frac{1}{2}\times\frac{\pi}{2} = \frac{\pi R^4}{4}.$$

Therefore, we have

$$\oint \mathbf{v}\cdot d\mathbf{a} = \frac{R^4}{4} - \frac{R^4}{4} + 0 + \frac{\pi R^4}{4} = \frac{\pi R^4}{4} \tag{5}$$

Comparing equations (4) and (5), we verify the divergence theorem.