Department of Physics
IIT Kanpur, Semester II, 2017-18

PHY103A: Physics II **Solution # 2** **Instructors: AKJ & SC**

Solution 2.1: Divergence theorem in cylindrical coordinates

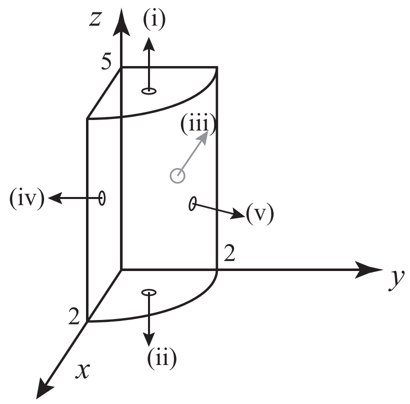

FIG. 1:

(a) Let us write the field as $\mathbf{V} = V_s\hat{\mathbf{s}} + V_\phi\hat{\phi} + V_z\hat{\mathbf{z}}$. The formula for divergence in the cylindrical coordinates can then be written as:

$$\boldsymbol{\nabla}\cdot\mathbf{V} = \frac{1}{s}\frac{\partial}{\partial s}(sV_s) + \frac{1}{s}\frac{\partial V_\phi}{\partial\phi} + \frac{\partial V_z}{\partial z}$$

Using the above formula, we get

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{V} &= \frac{1}{s}\frac{\partial}{\partial s}\left(ss(2+\sin^2\phi)\right) + \frac{1}{s}\frac{\partial}{\partial\phi}(s\sin\phi\cos\phi) + \frac{\partial}{\partial z}(3z)\\ &= \frac{1}{s}2s(2+\sin^2\phi) + \frac{1}{s}s(\cos^2\phi - \sin^2\phi) + 3\\ &= 4 + 2\sin^2\phi + \cos^2\phi - \sin^2\phi + 3\\ &= 4 + \sin^2\phi + \cos^2\phi + 3 = 8 \end{aligned} \tag{1}$$

(b) The divergence theorem that we need to verify in this case is

$$\int_{vol}(\boldsymbol{\nabla}\cdot\mathbf{V})d\tau = \oint_{surf}\mathbf{V}\cdot d\mathbf{a}.$$

Let us first calculate the volume integral

$$\int(\boldsymbol{\nabla}\cdot\mathbf{V})d\tau = \int_{s=0}^{2}\int_{\phi=0}^{\pi/2}\int_{z=0}^{5} 8s\,ds\,d\phi\,dz = 8\times 2\times\frac{\pi}{2}\times 5 = 40\pi$$

The surface integral has contributions due to the five surfaces shown in Fig. (**??**).

(i) $z = 5$; $d\mathbf{a} = sdsd\phi\hat{\mathbf{z}}$; $\mathbf{V}\cdot d\mathbf{a} = 3zsdsd\phi = 15sdsd\phi$; $\int\mathbf{V}\cdot d\mathbf{a} = 15\int_{s=0}^{2}\int_{\phi=0}^{\pi/2} sdsd\phi = 15\pi$

(ii) $z = 0$; $d\mathbf{a} = -sdsd\phi\hat{\mathbf{z}}$; $\mathbf{V}\cdot d\mathbf{a} = -3zsdsd\phi = 0$; $\int\mathbf{V}\cdot d\mathbf{a} = 0$

(iii) $\phi = \pi/2$; $d\mathbf{a} = dsdz\hat{\phi}$; $\mathbf{V}\cdot d\mathbf{a} = s\sin\phi\cos\phi dsdz = 0$; $\int\mathbf{V}\cdot d\mathbf{a} = 0$

(iv) $\phi = 0$; $d\mathbf{a} = -dsdz\hat{\phi}$; $\mathbf{V}\cdot d\mathbf{a} = -s\sin\phi\cos\phi dsdz = 0$; $\int\mathbf{V}\cdot d\mathbf{a} = 0$

(v) $s = 2$; $d\mathbf{a} = sd\phi dz\hat{\mathbf{s}}$; $\mathbf{V}\cdot d\mathbf{a} = s(2+\sin^2\phi)sd\phi dz = 4(2+\sin^2\phi)d\phi dz$; $\int \mathbf{V}\cdot d\mathbf{a} = 4\int_{\phi=0}^{\pi/2}\int_{z=0}^{5}(2+\sin^{\phi})d\phi dz = 4\times(\pi+\pi/4)\times 5 = 25\pi$

Therefore, adding all the five contribution, we get

$$\oint \mathbf{V}\cdot d\mathbf{a} = 15\pi + 25\pi = 40\pi \tag{2}$$

Thus, comparing Eqs. **??** and **??**, we have verified the divergence theorem.

(c) The formula for curl in the cylindrical coordinate is

$$\nabla\times\mathbf{V} = \left[\frac{1}{s}\frac{\partial V_z}{\partial\phi} - \frac{\partial V_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial V_s}{\partial z} - \frac{\partial V_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sV_\phi) - \frac{\partial V_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

The curl for the given vector is then

$$\begin{aligned}\nabla\times\mathbf{V} &= \left[\frac{1}{s}\frac{\partial}{\partial\phi}(3z) - \frac{\partial}{\partial z}(s\sin\phi\cos\phi)\right]\hat{\mathbf{s}} + \left[\frac{\partial}{\partial z}s(2+\sin^2\phi) - \frac{\partial}{\partial s}(3z)\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s^2\sin\phi\cos\phi) - \frac{\partial}{\partial\phi}(s(2+\sin^2\phi))\right]\hat{\mathbf{z}} \\ &= \frac{1}{s}(2s\sin\phi\cos\phi - s2\sin\phi\cos\phi)\hat{\mathbf{z}} \\ &= 0\end{aligned}$$

Solution 2.2: Applications of the Dirac delta function

(a) The electric charge density $\rho(\mathbf{r})$ of a point charge q at $\mathbf{r}'$ can be written as $\rho(\mathbf{r}) = q\delta^3(\mathbf{r}-\mathbf{r}')$. This can be verified by taking the volume integral: $\int\rho(\mathbf{r})d\tau = q\int\delta^3(\mathbf{r}-\mathbf{r}')d\tau = q$.

(b) The electric charge density in this case can be written as: $\rho(\mathbf{r}) = q\delta^3(\mathbf{r}-\mathbf{a}) - q\delta^3(\mathbf{r})$

(c) The electric charge density of a uniform, infinitesimally thin spherical shell of radius R and total charge Q is $\rho(\mathbf{r}) = A\delta(r-R)$. Please note this is just a one-dimensional Dirac delta function. In the two other dimensions, ϕ and z, the charge density is not zero. We find A by requiring that the total charge is Q, that is,

$$\int\rho d\tau = A\iiint\delta(r-R)r^2\sin\theta dr d\theta d\phi = A4\pi\int_{r=0}^{\infty}\delta(r-R)r^2dr = A\times 4\pi R^2 = Q$$

Therefore, we have $A = \dfrac{Q}{4\pi R^2}$.

Solution 2.3: Calculating charge density given an Electric field

To find the charge density from a given field, we need to use the divergence theorem: $\nabla\cdot\mathbf{E} = \rho/\epsilon_0$. For the given field we have

$$\nabla\cdot\mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2(A/r)) + 0 + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{B\sin\theta\cos\phi}{r}\right) = \frac{A}{r^2} + \frac{B}{r^2}(-\sin\phi) = \frac{1}{r^2}(A - B\sin\phi).$$

Therefore, the charge density is given by $\rho = \dfrac{\epsilon_0}{r^2}(A - B\sin\phi)$

Solution 2.4: Physical Electrostatic field

To determine whether a field is physical we need to check if $\nabla\times\mathbf{E} = 0$.

(a)

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix} = (0-2y)\hat{\mathbf{x}} + (0-3z)\hat{\mathbf{y}} + (0-x)\hat{\mathbf{z}} \neq 0$$

(b)

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & 2xy+z^2 & 2yz \end{vmatrix} = (2z-2z)\hat{\mathbf{x}} + (0-0)\hat{\mathbf{y}} + (2y-2y)\hat{\mathbf{z}} = 00$$

Thus, we find that only the second field can be a physical electrostatic field.

Solution 2.5: Calculating electric field for a given charge distribution

(a) Let us draw the Gaussian surface as shown in Fig. **??**(a). We thus have

$$\oint \mathbf{E} \cdot d\mathbf{a} = E \times 4\pi r^2 = \frac{Q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0}\frac{4}{3}\pi r^3 k$$

Therefore, we have

$$\mathbf{E} = \frac{1}{3\epsilon_0} k r \hat{\mathbf{r}}$$

(b) We calculate the electric field in the three regions by drawing appropriate Gaussian surfaces and then using Gauss's law:

(i) $Q_{enc} = 0$. So $\mathbf{E} = 0$

(ii)

$$\oint \mathbf{E} \cdot d\mathbf{a} = E \times 4\pi r^2; \qquad \frac{Q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0}\int \rho d\tau = \frac{1}{\epsilon_0}\int_{r'=a}^{r}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{k}{r'^2} r'^2 \sin\theta dr d\theta d\phi = \frac{4\pi k}{\epsilon_0}(r-a)$$

Therefore, $\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{enc}}{\epsilon_0} \quad \Rightarrow \quad \mathbf{E} = \frac{k}{\epsilon_0}\left(\frac{r-a}{r^2}\right)\hat{\mathbf{r}}.$

(iii)

$$\oint \mathbf{E} \cdot d\mathbf{a} = E \times 4\pi r^2 \qquad \frac{Q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0}\int \rho d\tau = \frac{1}{\epsilon_0}\int_{r'=a}^{b}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{k}{r'^2} r'^2 \sin\theta dr d\theta d\phi = \frac{4\pi k}{\epsilon_0}(b-a).$$

Therefore, $\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{enc}}{\epsilon_0} \quad \Rightarrow \quad \mathbf{E} = \frac{k}{\epsilon_0}\left(\frac{b-a}{r^2}\right)\hat{\mathbf{r}}.$

(c) As shown in part (a), the field inside a uniformly charged sphere of charge density ρ is given by $\mathbf{E} = \frac{\rho r}{3\epsilon_0}\hat{\mathbf{r}} = \frac{\rho}{3\epsilon_0}\mathbf{r}$. Let us take $\mathbf{r}_+$ and $\mathbf{r}_-$ to be the radius vectors to a point in the overlap region from the centers of the positively and negatively charged sphere, respectively. The electric field in the overlap region is equal to the sum of the electric fields due to the positively and negatively charged spheres. Thus the electric field E_{overlap} in the overlap region is [See Fig. **??**(c)]

$$E_{\text{overlap}} = \frac{\rho}{3\epsilon_0}\mathbf{r}_+ - \frac{\rho}{3\epsilon_0}\mathbf{r}_- = \frac{\rho}{3\epsilon_0}(\mathbf{r}_+ - \mathbf{r}_+) = \frac{\rho}{3\epsilon_0}\mathbf{d}$$

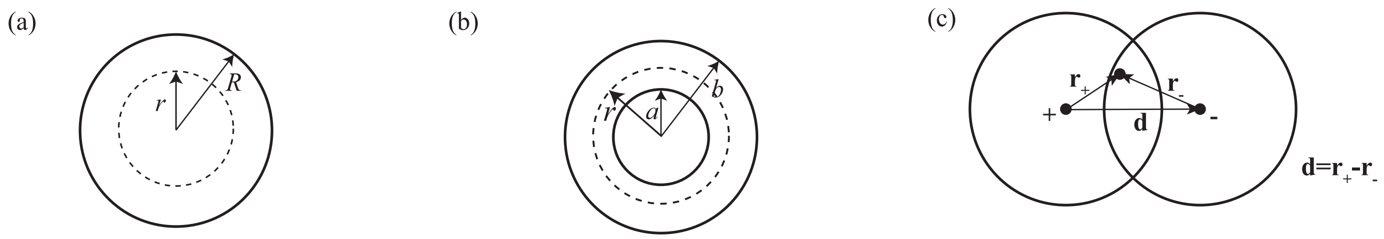

FIG. 2:

Solution 2.6: Scalar and Vector Potentials

(a) The curl of a vector $\mathbf{v}$ is defined as

$$\boldsymbol{\nabla} \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{z}}$$

Using this formula, we get,

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{v_1} &= (0 - 6xz)\hat{\mathbf{x}} + (0 + 2z)\hat{\mathbf{y}} + (3z^2 - 0)\hat{\mathbf{z}} \\ \boldsymbol{\nabla} \times \mathbf{v_2} &= (0 - 2y)\hat{\mathbf{x}} + (0 - 3z)\hat{\mathbf{y}} + (0 - x)\hat{\mathbf{z}} \\ \boldsymbol{\nabla} \times \mathbf{v_3} &= (2z - 2z)\hat{\mathbf{x}} + (0 - 0)\hat{\mathbf{y}} + (2y - 2y)\hat{\mathbf{z}} = 0 \end{aligned}$$

Therefore the vector $\mathbf{v_3}$ can be expressed as the gradient of a scalar.

(b) The divergence of a vector $\mathbf{v_3}$ is defined as

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Using this formula, we get

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{v_1} &= 2x - 2x + 0 = 0 \\ \boldsymbol{\nabla} \cdot \mathbf{v_2} &= y + 2z + 3x \\ \boldsymbol{\nabla} \cdot \mathbf{v_3} &= 0 + 2x + 2y \end{aligned}$$

Therefore the vector $\mathbf{v_1}$ can be expressed as the curl of a vector $\mathbf{A}$, that is, $\mathbf{v_1} = \boldsymbol{\nabla} \times \mathbf{A}$. We write the vector as $\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$. Therefore, we have

$$\begin{aligned} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} &= x^2 \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} &= 3z^2 x \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} &= -2xz \end{aligned}$$

Choosing $A_x = 0$, we get

$$\begin{aligned} \frac{\partial A_z}{\partial x} &= -3z^2 x \Rightarrow A_z = -\frac{3}{2}x^2 z^2 + f(y,z) \\ \frac{\partial A_y}{\partial x} &= -2xz \Rightarrow A_y = -x^2 z + g(y,z) \\ \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} &= \frac{\partial f}{\partial y} + x^2 - \frac{\partial g}{\partial z} = x^2 \quad \Rightarrow \frac{\partial f}{\partial y} - \frac{\partial g}{\partial z} = 0 \end{aligned}$$

We may even choose $f = g = 0$, and thus the vector is $\mathbf{A} = -x^2 z\hat{\mathbf{y}} - \frac{3}{2}x^2 z^2 \hat{\mathbf{z}}$. This vector is not unique. Any gradient can be added to this vector without changing the curl

Solution 2.7: Electric Potential

Let us consider the thin stripe on the hemisphere as shown in Fig. ??. Every point on this stripe is at equal distance (R) from the center. Therefore, the potential V_{center} at the center can be written as

$$V_{\text{center}} = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{R} da = \frac{1}{4\pi\epsilon_0}\frac{\sigma}{R}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} R^2 \sin\theta d\theta d\phi = \frac{1}{4\pi\epsilon_0}\frac{\sigma}{R}2\pi R^2 = \frac{\sigma R}{2\epsilon_0}$$

The potential V_{center} at pole can be written as

$$V_{\text{pole}} = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{r} da = \frac{1}{4\pi\epsilon_0}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}\frac{\sigma}{r}R^2 \sin\theta d\theta d\phi = \frac{\sigma 2\pi R^2}{4\pi\epsilon_0}\int_{\theta=0}^{\pi/2}\frac{1}{r}\sin\theta d\theta$$

We have $r = \sqrt{R^2 + R^2 - 2R^2\cos\theta} = \sqrt{2}R\sqrt{1-\cos\theta}$. Therefore,

$$V_{\text{pole}} = \frac{\sigma 2\pi R^2}{4\pi\epsilon_0\sqrt{2}R}\int_{\theta=0}^{\pi/2}\frac{\sin\theta}{\sqrt{1-\cos\theta}}d\theta = \frac{\sigma R}{2\sqrt{2}\epsilon_0}(2\sqrt{1-\cos\theta})\Big|_0^{\pi/2} = \frac{\sigma R}{\sqrt{2}\epsilon_0}(1-0) = \frac{\sigma R}{\sqrt{2}\epsilon_0}$$

Thus $V_{\text{pole}} - V_{\text{center}} = \frac{\sigma R}{2\epsilon_0}(\sqrt{2}-1)$.

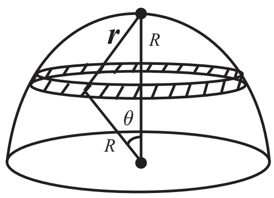

FIG. 3: