Department of Physics
IIT Kanpur, Semester II, 2017-18

PHY103A: Physics II | **Solution # 3** | **Instructors: AKJ & SC**

Solution 3.1: Finding potential, given a charge distribution (Griffiths 3rd ed. Prob 2.26)

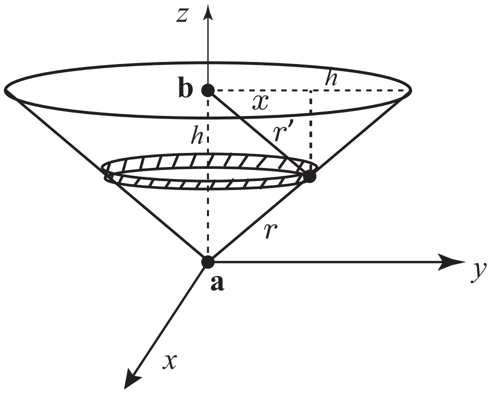

FIG. 1:

The potential $V(\mathbf{a})$ at point $\mathbf{a}$ is given by (see Fig. 1)

$$V(\mathbf{a}) = \frac{1}{4\pi\epsilon_0}\int_0^{\sqrt{2}h}\left(\frac{\sigma 2\pi x}{r}\right)dr$$

we have $x = r/\sqrt{2}$. Therefore we get,

$$V(\mathbf{a}) = \frac{2\pi\sigma}{4\pi\epsilon_0}\int_0^{\sqrt{2}h}\left(\frac{1}{\sqrt{2}}\right)dr = \frac{2\pi\sigma}{4\pi\epsilon_0}\frac{1}{\sqrt{2}}(\sqrt{2}h) = \frac{\sigma h}{2\epsilon_0}$$

The potential $V(\mathbf{b})$ at point $\mathbf{b}$ is given by (see Fig. 1)

$$V(\mathbf{b}) = \frac{1}{4\pi\epsilon_0}\int_0^{\sqrt{2}h}\left(\frac{\sigma 2\pi x}{r'}\right)dr$$

we have $x = r/\sqrt{2}$ and $r' = \sqrt{h^2 + r^2 - \sqrt{2}hr}$. Therefore we get,

$$\begin{aligned}
V(\mathbf{b}) =& \frac{1}{4\pi\epsilon_0}\int_0^{\sqrt{2}h}\left(\frac{\sigma 2\pi x}{r'}\right)dr = \frac{2\pi\sigma}{4\pi\epsilon_0}\frac{1}{\sqrt{2}}\int_0^{\sqrt{2}h}\left(\frac{r}{\sqrt{h^2+r^2-\sqrt{2}hr}}\right)dr \\
=& \frac{\sigma}{2\sqrt{2}\epsilon_0}\left[\sqrt{h^2+r^2-\sqrt{2}hr} + \frac{h}{\sqrt{2}}\ln\left(2\sqrt{h^2+r^2-\sqrt{2}hr}+2r-\sqrt{2}h\right)\right]\Big|_0^{\sqrt{2}h} \\
=& \frac{\sigma}{2\sqrt{2}\epsilon_0}\left[h + \frac{h}{\sqrt{2}}\ln(2h+2\sqrt{2}h-\sqrt{2}h) - h - \frac{h}{\sqrt{2}}\ln(2h-\sqrt{2}h)\right] \\
=& \frac{\sigma}{2\sqrt{2}\epsilon_0}\frac{h}{\sqrt{2}}\left[\ln(2h+\sqrt{2}h)-\ln(2h-\sqrt{2}h)\right] = \frac{\sigma h}{4\epsilon_0}\ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right) = \frac{\sigma h}{4\epsilon_0}\ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right) \\
=& \frac{\sigma h}{4\epsilon_0}\ln\left(\frac{(2+\sqrt{2})^2}{2}\right) = \frac{\sigma h}{2\epsilon_0}\ln\left(1+\sqrt{2}\right)
\end{aligned}$$

Thus we get the required potential difference to be

$$V(\mathbf{a}) - V(\mathbf{b}) = \frac{\sigma h}{2\epsilon_0}\left[1 - \ln\left(1+\sqrt{2}\right)\right]$$

Solution 3.2: Finding field and charge density, given an electric potential, (Griffiths 3rd ed. Prob 2.46)

(a) The electric potential is $V(\mathbf{r}) = A\dfrac{e^{-\lambda r}}{r}$. Therefore, the electric field $\mathbf{E}(\mathbf{r})$ can be written as

$$\mathbf{E} = -\boldsymbol{\nabla} V = -A\frac{\partial}{\partial r}\left(\frac{e^{-\lambda r}}{r}\right)\hat{\mathbf{r}} = -A\left[\frac{r(-\lambda)e^{-\lambda r} - e^{-\lambda r}}{r^2}\right]\hat{\mathbf{r}}$$
$$= Ae^{-\lambda r}(1+\lambda r)\frac{\hat{\mathbf{r}}}{r^2}$$

(b) The corresponding charge density $\rho(r)$ can be calculated by using the differential form of Gauss's Law $\rho = \epsilon_0 \boldsymbol{\nabla}\cdot\mathbf{E}$. Using the product rule for divergence, $\boldsymbol{\nabla}\cdot(f\mathbf{A}) = f(\boldsymbol{\nabla}\cdot\mathbf{A}) + \mathbf{A}\cdot(\boldsymbol{\nabla} f)$, we obtain

$$\rho = \epsilon_0\boldsymbol{\nabla}\cdot\mathbf{E} = \epsilon_0 Ae^{-\lambda r}(1+\lambda r)\boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) + \epsilon_0 A\frac{\hat{\mathbf{r}}}{r^2}\cdot\boldsymbol{\nabla}\left[e^{-\lambda r}(1+\lambda r)\right]$$

Next we use the properties of the Dirac-delta function and the formula for gradient in spherical coordinates to get

$$\epsilon_0 Ae^{-\lambda r}(1+\lambda r)\boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = \epsilon_0 Ae^{-\lambda r}(1+\lambda r)4\pi\delta^3(\mathbf{r}) = \epsilon_0 A4\pi\delta^3(\mathbf{r})$$
$$\epsilon_0 A\frac{\hat{\mathbf{r}}}{r^2}\cdot\boldsymbol{\nabla}\left[e^{-\lambda r}(1+\lambda r)\right] = \epsilon_0 A\frac{\hat{\mathbf{r}}}{r^2}\cdot\frac{\partial}{\partial r}\left[e^{-\lambda r}(1+\lambda r)\right]\hat{\mathbf{r}}$$
$$= \epsilon_0 A\frac{\hat{\mathbf{r}}}{r^2}\cdot\left[-\lambda e^{-\lambda r}(1+\lambda r) + e^{-\lambda r}\lambda\right]\hat{\mathbf{r}}$$
$$= \epsilon_0 A\frac{\hat{\mathbf{r}}}{r^2}\cdot\left[-\lambda^2 re^{-\lambda r}\right]\hat{\mathbf{r}}$$
$$= -\epsilon_0 A\frac{\lambda^2}{r}e^{-\lambda r}$$

Therefore, we get the charge density $\rho(r)$ as

$$\rho = \epsilon_0 A\left[4\pi\delta^3(\mathbf{r}) - \frac{\lambda^2}{r}e^{-\lambda r}\right]$$

(c) The total charge Q can now be calculated to be

$$Q = \int \rho d\tau$$
$$= \epsilon_0 A4\pi\int\delta^3(\mathbf{r})d\tau - \epsilon_0 A\lambda^2\int_0^\infty \frac{e^{-\lambda r}}{r}4\pi r^2 dr$$
$$= \epsilon_0 A4\pi - \epsilon_0 A\lambda^2 4\pi\int_0^\infty re^{-\lambda r}dr$$
$$= \epsilon_0 A4\pi - \epsilon_0 A\lambda^2 4\pi\left(\frac{1}{\lambda^2}\right)$$
$$= 0$$

Therefore the total charge is zero.

Solution 3.3: Verifying Poisson's Equation (Griffiths 3rd ed. Prob 2.29)

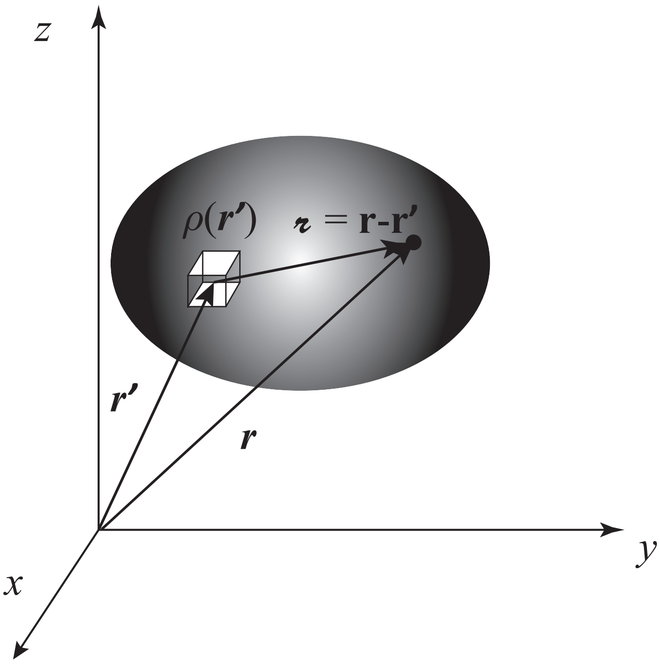

FIG. 2:

The potential $V(\mathbf{r})$ at $\mathbf{r}$ due to the localized charge distribution is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau',$$

We note that the charge distribution has been represented in the $(\mathbf{r}', \theta', \phi')$ coordinates. We take the Laplacian of the potential in $(\mathbf{r}, \theta, \phi)$ coordinates. Therefore, we get

$$\begin{aligned}
\nabla^2 V(\mathbf{r}) &= \nabla^2 \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau' \\
&= \frac{1}{4\pi\epsilon_0}\int \nabla^2 \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau' \\
&= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}') \left(\nabla^2 \frac{1}{\mathscr{r}}\right) d\tau' \\
&= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}') \left(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \frac{1}{\mathscr{r}}\right) d\tau' \\
&= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}') \left(\boldsymbol{\nabla} \cdot \frac{-\hat{\mathscr{r}}}{\mathscr{r}^2}\right) d\tau' \\
&= \frac{1}{4\pi\epsilon_0}\int -\rho(\mathbf{r}') 4\pi \delta^3(\mathscr{r}) d\tau' \\
&= \frac{1}{4\pi\epsilon_0}\int -\rho(\mathbf{r}') 4\pi \delta^3(\mathbf{r} - \mathbf{r}') d\tau' \\
&= -\frac{1}{\epsilon_0}\rho(\mathbf{r})
\end{aligned}$$

Thus, we see that the given potential satisfies the Poisson's equation.

Solution 3.4: Electrostatic energy of two spherical shells (Griffiths 3rd ed. Prob 2.34)

(a) The electric field due to the two shells is given by $\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$, for $(a < r < b)$, and $\mathbf{E}(\mathbf{r}) = 0$, otherwise. The energy of this configuration is:

$$W = \frac{\epsilon_0}{2}\int E^2 d\tau = \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2 \int_a^b \left(\frac{1}{r^2}\right)^2 4\pi r^2 dr = \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

(b) Let's us first calculate the energy of the individual shells. The electric filed due to the shell of radius a is $\mathbf{E}_a(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$, for $(r > a)$, and $\mathbf{E}_a(\mathbf{r}) = 0$, otherwise. The electric filed due to the shell of radius b is $\mathbf{E}_b(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$, for $(r > b)$, and $\mathbf{E}_b(\mathbf{r}) = 0$, otherwise. Therefore, the energy of the first spherical shell is:

$$W_a = \frac{\epsilon_0}{2}\int E_a^2 d\tau = \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2 \int_a^\infty \left(\frac{1}{r^2}\right)^2 4\pi r^2 dr = \frac{q^2}{8\pi\epsilon_0 a}$$

Similarly, the energy of the second spherical shell is

$$W_b = \frac{\epsilon_0}{2}\int E_b^2 d\tau = \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2 \int_b^\infty \left(\frac{1}{r^2}\right)^2 4\pi r^2 dr = \frac{q^2}{8\pi\epsilon_0 b}$$

The interaction energy of this system is therefore:

$$\begin{aligned} W_{\text{int}} &= \epsilon_0 \int \mathbf{E_a}\cdot\mathbf{E_b} d\tau = W - W_a - W_b = -\frac{q^2}{8\pi\epsilon_0 b} - \frac{q^2}{8\pi\epsilon_0 b} \\ &= -\frac{q^2}{4\pi\epsilon_0 b} \end{aligned}$$

Solution 3.5: Electrostatic Force

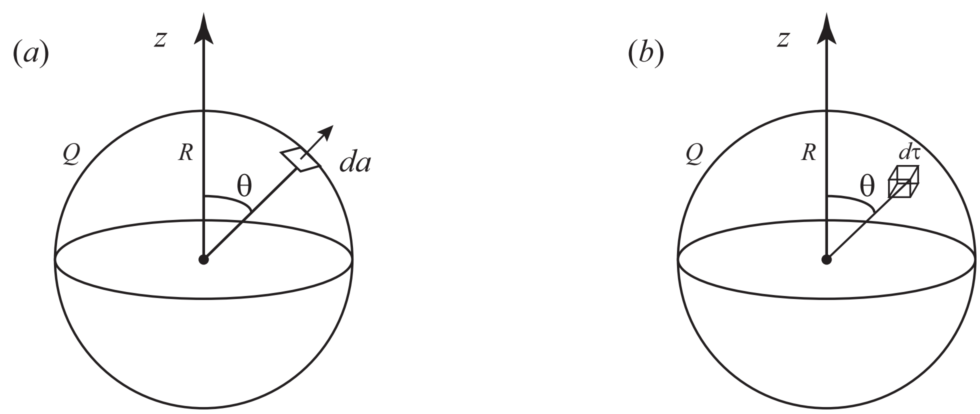

FIG. 3:

(a) The electric filed due to the metal sphere of radius R is given by $\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}$, for $(r \geq R)$, and $\mathbf{E}(\mathbf{r}) = 0$, otherwise. From the symmetry of the problem, it is clear that the total electrostatic force on northern hemisphere will be in the z direction. Now, the electrostatic force per unit area in the z-direction at the area element da, as shown in Fig. 3(a), is:

$$f_z = \sigma \mathbf{E}_{\text{other}} \cdot \hat{\mathbf{z}} = \sigma \frac{\mathbf{E}(\mathbf{r})}{2} \cdot \hat{\mathbf{z}} = \frac{Q}{4\pi R^2} \frac{1}{2} \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \cos\theta = \frac{Q^2}{32\pi^2\epsilon_0 R^4} \cos\theta$$

Therefore, the total repulsive force on the northern hemisphere is

$$\begin{aligned} F_z = \int f_z da &= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \left(\frac{Q^2}{32\pi^2\epsilon_0 R^4} \cos\theta \right) R^2 \sin\theta d\theta d\phi \\ &= \frac{Q^2}{32\pi^2\epsilon_0 R^2} 2\pi \int_{\theta=0}^{\pi/2} \cos\theta \sin\theta d\theta \\ &= \frac{Q^2}{32\pi^2\epsilon_0 R^2} 2\pi \frac{1}{2} \\ &= \frac{Q^2}{32\pi\epsilon_0 R^2} \end{aligned}$$

(b) The electric filed inside a uniformly charged sphere of radius R and charge Q is given by $\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{Qr}{R^3} \hat{\mathbf{r}}$. From the symmetry of the problem, it is clear that the total electrostatic force on the northern hemisphere will be in the z direction. Now, the electrostatic force per unit volume in the z-direction on the volume element $d\tau$, as shown in Fig. 3(b), is:

$$f_z = \rho \mathbf{E}(\mathbf{r}) \cdot \hat{\mathbf{z}} = \frac{3Q}{4\pi R^3} \frac{1}{4\pi\epsilon_0} \frac{Qr}{R^3} \cos\theta = \frac{3Q^2}{16\pi^2\epsilon_0 R^6} r \cos\theta$$

Therefore, the electrostatic force on the northern hemisphere is

$$\begin{aligned} F_z = \int f_z d\tau &= \int_0^R \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \left(\frac{3Q^2}{16\pi^2\epsilon_0 R^6} r \cos\theta \right) r^2 \sin\theta d\theta d\phi \\ &= \frac{3Q^2}{16\pi^2\epsilon_0 R^6} \int_0^R \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} r^3 \cos\theta \sin\theta d\theta d\phi \\ &= \frac{3Q^2}{16\pi^2\epsilon_0 R^6} \int_0^R r^3 dr \int_{\theta=0}^{\pi/2} \cos\theta \sin\theta d\theta \int_{\phi=0}^{2\pi} d\phi \\ &= \frac{3Q^2}{16\pi^2\epsilon_0 R^6} \times \frac{R^4}{4} \times \frac{1}{2} \times 2\pi \\ &= \frac{3Q^2}{64\pi\epsilon_0 R^2} \end{aligned}$$

Solution 3.6: Capacitance of coaxial metal cylinders (Griffiths 3rd ed. Prob 2.39)

Suppose that for a length L, the charge on the inner cylinder is Q and the charge on the outer cylinder is $-Q$. Using the Gaussian surface as shown in Fig. 4, it can be shown that the field in between the cylinders is $\mathbf{E}(\mathbf{s}) = \frac{Q}{2\pi\epsilon_0 L} \frac{1}{s} \hat{\mathbf{s}}$. The potential difference between the cylinders is therefore,

$$V(b) - V(a) = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = -\frac{Q}{2\pi\epsilon_0 L} \int_a^b \frac{1}{s} ds = -\frac{Q}{2\pi\epsilon_0 L} \ln\left(\frac{b}{a}\right).$$

We see that a is at a higher potential. So, we take the potential difference as $V = V(a) - V(b) = \frac{Q}{2\pi\epsilon_0 L} \ln\left(\frac{b}{a}\right)$. The capacitance C of this configuration is therefore given by

$$C = \frac{Q}{V} = \frac{2\pi\epsilon_0}{\ln\left(\dfrac{b}{a}\right)}$$

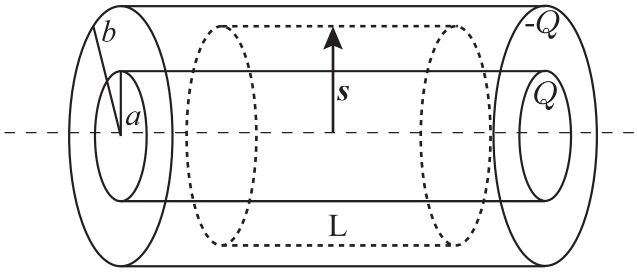

FIG. 4: