Department of Physics
IIT Kanpur, Semester II, 2017-18

PHY103A: Physics II **Solution # 4** **Instructors: AKJ & SC**

Solution 4.1: Force with image charges (Griffiths 3rd ed. Prob 3.6)

As far as force is concerned, this problem is the same if we remove the grounded conducting plate and simply put an image charge $+2q$ at $z = -d$ and an image charge $-q$ at $z = -3d$. Therefore, the force on the charge $+q$ is given by

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\left[\frac{-2q}{(2d)^2} + \frac{2q}{(4d)^2} + \frac{-q}{(6d)^2}\right]\hat{\mathbf{z}} = \frac{q^2}{4\pi\epsilon_0 d^2}\left[\frac{-1}{2} + \frac{1}{8} + \frac{-1}{36}\right]\hat{\mathbf{z}} = -\frac{q^2}{4\pi\epsilon_0 d^2}\frac{29}{72}\hat{\mathbf{z}}$$

Solution 4.2: Infinite-line image charge (Griffiths 3rd ed. Prob 3.9)

This problem is an extension of the problem in which a point charge is placed above an infinite grounded conducting plane. So, in order to obtain the correct potential, we need to put an infinite line charge $-\lambda$ running parallel to the x-axis at a distance $-d$ from the $x-y$ plane. (see Fig. 1).

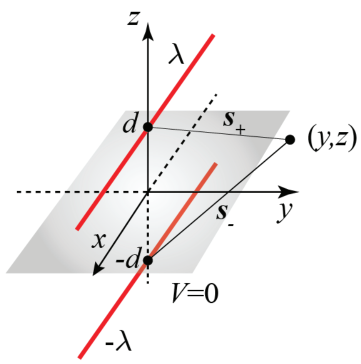

FIG. 1:

(a) Let's first find the potential at a point (y, z) on the $y-z$ plane. Assume that the distance of the point (y, z) from the top infinite line charge is given by the magnitude of the vector $\hat{\mathbf{s}}_+$; similarly the distance of the point (y, z) from the bottom infinite line charge is given by the magnitude of the vector $\hat{\mathbf{s}}_-$. Taking the potential $V = 0$ at $x-y$ plane to be the reference, the potential due to top infinite line charge can be shown to be $V_+ = -\frac{2\lambda}{4\pi\epsilon_0}\ln\left(\frac{s_+}{d}\right)$. Similarly, the potential due to the bottom infinite line charge can be shown to be $V_- = +\frac{2\lambda}{4\pi\epsilon_0}\ln\left(\frac{s_-}{d}\right)$. Therefore the total potential at (y, z) is given by

$$\begin{aligned}
V(y,z) &= -\frac{2\lambda}{4\pi\epsilon_0}\ln\left(\frac{s_+}{d}\right) + \frac{2\lambda}{4\pi\epsilon_0}\ln\left(\frac{s_-}{d}\right) \\
&= \frac{2\lambda}{4\pi\epsilon_0}\ln\left(\frac{s_-}{s_+}\right) \\
&= \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{s_-^2}{s_+^2}\right) \\
&= \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right]
\end{aligned}$$

(b) The induced charge density is given by $\sigma = -\epsilon_0 \dfrac{\partial V}{\partial n}$, where $\hat{\mathbf{n}}$ is the vector in the direction perpendicular to the surface. In the present problem, $\hat{\mathbf{n}} = \hat{\mathbf{z}}$. Therefore,

$$
\begin{aligned}
\sigma &= -\epsilon_0 \frac{\partial V}{\partial n} = -\epsilon_0 \frac{\partial V}{\partial z}\Big|_{z=0} \\
&= -\epsilon_0 \frac{\partial}{\partial z}\left[\frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right]\right]\Big|_{z=0} \\
&= -\epsilon_0 \frac{\lambda}{4\pi\epsilon_0}\left[\frac{2(z+d)}{y^2+(z+d)^2} - \frac{2(z-d)}{y^2+(z-d)^2}\right]\Big|_{z=0} \\
&= -\epsilon_0 \frac{\lambda}{4\pi\epsilon_0}\left[\frac{2d}{y^2+z^2} + \frac{2d}{y^2+d^2}\right] \\
&= -\frac{\lambda d}{\pi(y^2+d^2)^2}
\end{aligned}
$$

Solution 4.3: Far-field potential (Griffiths 3rd ed. Prob 3.26)

In this problem, a point $\mathbf{r}$ on the sphere is represented in the (r, θ, ϕ) coordinates. An observation point $\mathbf{r_0}$ is represented in the (r_0, θ_0, ϕ_0) coordinates. The angle between the vectors $\mathbf{r}$ and $\mathbf{r_0}$ is represented by α. So, the formula for multipole expansion of potential V can be written as

$$
V(\mathbf{r_0}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r_0^{n+1}}\int r^n P_n(\cos\alpha)\rho(\mathbf{r})d\tau
$$

Now, since we are interested in observation points along z-axis only, the angle α between the two vectors in essentially the angle θ. Therefore we can write the above expression as

$$
V(\mathbf{r_0}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\int r^n P_n(\cos\theta)\rho(\mathbf{r})d\tau
$$

Let's calculate the above potential term by term in the limit $z \gg R$.

1. **Monopole term:** We have

$$
\begin{aligned}
V_{\text{mono}}(\mathbf{r_0}) &= \frac{1}{4\pi\epsilon_0 z}\int \rho(\mathbf{r})d\tau \\
&= \frac{1}{4\pi\epsilon_0 z}\int\left[k\frac{R}{r^2}(R-2r)\sin\theta\right] r^2\sin\theta dr d\theta d\phi \\
&= \frac{1}{4\pi\epsilon_0 z}kR\int(R-2r)\sin^2\theta dr d\theta d\phi
\end{aligned}
$$

The r-integral yields $\int_0^R(R-2r)dr = (Rr - r^2)\Big|_0^R = R^2 - R^2 = 0$. Therefore, the monopole potential is zero:

$$
V_{\text{mono}}(\mathbf{r_0}) = \frac{1}{4\pi\epsilon_0 z}kR\int(R-2r)\sin^2\theta dr d\theta d\phi = 0
$$

2. **Dipole term:** We have

$$\begin{aligned} V_{\text{dip}}(\mathbf{r_0}) &= \frac{1}{4\pi\epsilon_0 z^2}\int rP_1(\cos\theta)\rho(\mathbf{r})d\tau \\ &= \frac{1}{4\pi\epsilon_0 z^2}\int r\cos\theta\rho(\mathbf{r})d\tau \\ &= \frac{1}{4\pi\epsilon_0 z^2}\int r\cos\theta\left[k\frac{R}{r^2}(R-2r)\sin\theta\right]r^2\sin\theta drd\theta d\phi \end{aligned}$$

The θ-integral yields $\int_0^\pi \sin^2\theta\cos\theta d\theta = \frac{\sin^3\theta}{3}\Big|_0^\pi = \frac{1}{3}(0-0) = 0$. Therefore, the dipole potential is also zero:

$$V_{\text{dip}}(\mathbf{r_0}) = \frac{1}{4\pi\epsilon_0 z^2}\int r\cos\theta\left[k\frac{R}{r^2}(R-2r)\sin\theta\right]r^2\sin\theta drd\theta d\phi = 0$$

3. **Quadrupole term:** We have

$$\begin{aligned} V_{\text{quad}}(\mathbf{r_0}) &= \frac{1}{4\pi\epsilon_0 z^3}\int r^2P_2(\cos\theta)\rho(\mathbf{r})d\tau \\ &= \frac{1}{4\pi\epsilon_0 z^3}\int r^2(3\cos^2\theta-1)/2\rho(\mathbf{r})d\tau \\ &= \frac{1}{8\pi\epsilon_0 z^3}\int r^2(3\cos^2\theta-1)\left[k\frac{R}{r^2}(R-2r)\sin\theta\right]r^2\sin\theta drd\theta d\phi \end{aligned}$$

The three integrals can be calculated as follows:

$$\begin{aligned} &r-\text{integral}: \int_0^R r^2(R-2r)dr = \left(\frac{r^3}{3}R-\frac{r^4}{2}\right)\Big|_0^R = -\frac{R^4}{6} \\ &\theta-\text{integral}: \int_0^\pi (3\cos^2\theta-1)\sin^2\theta d\theta = 2\int_0^\pi \sin^2\theta d\theta - 3\int_0^\pi \sin^4\theta d\theta = 2\left(\frac{\pi}{2}\right)-3\left(\frac{3\pi}{8}\right) = \pi\left(1-\frac{9}{8}\right) = -\frac{\pi}{8} \\ &\phi-\text{integral}: \int_0^\pi d\phi = 2\pi \end{aligned}$$

Therefore,

$$V_{\text{quad}}(\mathbf{r_0}) = \frac{kR}{8\pi\epsilon_0 z^3}\left(-\frac{R^4}{6}\right)\times\left(-\frac{\pi}{8}\right)\times(2\pi) = \frac{1}{4\pi\epsilon_0}\frac{k\pi^2R^5}{48z^3}$$

So, the approximate potential is the potential of the quadrupole:

$$V(\mathbf{r_0}) \approx V_{\text{quad}}(\mathbf{r_0}) = \frac{1}{4\pi\epsilon_0}\frac{k\pi^2R^5}{48z^3}$$

Solution 4.4: Potential due to a four-charge system (Griffiths 3rd ed. Prob 3.27)

The total charge of the system is zero. So, the monopole term in the potential would be zero. Now, let's calculate the dipole term [see Fig. 2]. We have

$$\mathbf{p} = (3qa-qa)\hat{\mathbf{z}} + (-2qa-2q(-a))\hat{\mathbf{y}} = 2qa\hat{\mathbf{z}}$$

Therefore,

$$V \approx V_{\text{dipole}} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{2qa\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{2qa\cos\theta}{r^2}$$

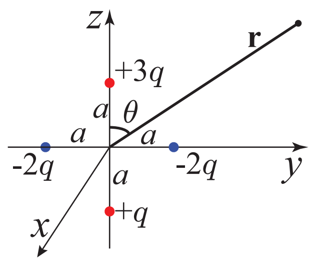

FIG. 2:

Solution 4.5: Far-field Potential due to a spherical-charge distribution (Griffiths 3rd ed. Prob 3.28)

(a) From the symmetry in the problem it is clear that the net dipole moment will be in the z direction. Therefore

$$\begin{aligned}\mathbf{p} &\equiv \int \mathbf{r}\rho(\mathbf{r})d\tau = \int z\hat{\mathbf{z}}\rho(\mathbf{r})d\tau = \int z\rho(\mathbf{r})d\tau\hat{\mathbf{z}} = \int z\sigma da\hat{\mathbf{z}} \\ &= \int (R\cos\theta)(k\cos\theta)R^2\sin\theta d\theta d\phi\hat{\mathbf{z}} \\ &= 2\pi R^3 k\int_0^\pi \cos^2\theta\sin\theta d\theta\hat{\mathbf{z}} \\ &= 2\pi R^3 k\left(-\frac{\cos^3\theta}{3}\right)\Big|_0^\pi\hat{\mathbf{z}} \\ &= \frac{4\pi R^3 k}{3}\hat{\mathbf{z}}\end{aligned}$$

(b) We find that the total charge of the system is zero. So, there will be no monopole contribution to the far-field potential. However since the dipole moment is not zero, the first contribution will be from the dipole term and the far-field potential can be written as

$$V \approx V_{\text{dipole}} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{4\pi R^3 k}{3}\frac{\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{4\pi R^3 k}{3}\frac{\cos\theta}{r^2} = \frac{kR^3}{3\epsilon_0}\frac{\cos\theta}{r^2}$$

Solution 4.6: Electric field of a pure dipole (Griffiths 3rd ed. Prob 3.33)

The electric field of a pure dipole is given by

$$\begin{aligned}\mathbf{E}_{\text{dip}}(\mathbf{r}) &= \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}) \\ &= \frac{1}{4\pi\epsilon_0 r^3}(3p\cos\theta\hat{\mathbf{r}} - p\cos\theta\hat{\mathbf{r}} + p\sin\theta\hat{\theta}) \\ &= \frac{1}{4\pi\epsilon_0 r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - (\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - (\mathbf{p}\cdot\hat{\theta})\hat{\theta}] \\ &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]\end{aligned}$$

Solution 4.7: Field and potential due to a three-charge system (Griffiths 3rd ed. Prob 3.32)

The total charge of the three-charge system is $-q$. Therefore the monopole contribution is not zero and is given by

$$V_{\rm mono} = \frac{1}{4\pi\epsilon_0}\frac{-q}{r} = -\frac{q}{4\pi\epsilon_0 r}$$

The dipole moment $\mathbf{p}$ of the charge system is $\mathbf{p} = qa\hat{\mathbf{z}} + [-qa - q(-a)]\hat{\mathbf{y}} = qa\hat{\mathbf{z}}$. So, the dipole contribution to the potential is

$$V_{\rm dip} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{qa\cos\theta}{r^2}$$

Therefore, the potential up to two terms in the multipole expansion is given by

$$V \approx V_{\rm mono} + V_{\rm dip} = \frac{q}{4\pi\epsilon_0}\left(-\frac{1}{r} + \frac{qa\cos\theta}{r^2}\right)$$

The electric field $\mathbf{E}$ is given by

$$\mathbf{E} = -\boldsymbol{\nabla} V \approx \frac{q}{4\pi\epsilon_0}\left[-\frac{1}{r^2}\hat{\mathbf{r}} + \frac{a}{r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}\right)\right]$$

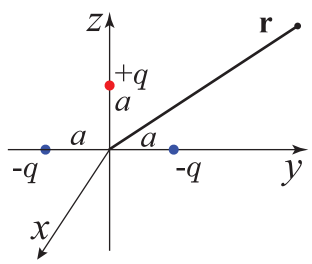

FIG. 3: