Department of Physics
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PHY103A: Physics II — Solution # 5 — **Instructors: AKJ & SC**

Solution 5.1: Force and Torque on a dipole (Griffiths 3rd ed., Prob 4.5 & Prob. 4.9)

(a) The field due to the dipole $\mathbf{p_1}$ is given by

$$\mathbf{E_1}(r,\theta) = \frac{p_1}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}).$$

The field due the dipole $\mathbf{p_1}$ at $\mathbf{p_2}$ is given by substituting $\theta = \pi/2$ in the above equation which yields

$$\mathbf{E_1}(r,\theta) = \frac{p_1}{4\pi\epsilon_0 r^3}\hat{\theta} = \frac{p_1}{4\pi\epsilon_0 r^3}(-\hat{\mathbf{z}})$$

So, the torque on $\mathbf{p_2}$ is

$$\mathbf{N_2} = \mathbf{p_2} \times \mathbf{E_1} = p_2(\hat{\mathbf{y}}) \times \frac{p_1}{4\pi\epsilon_0 r^3}(-\hat{\mathbf{z}}) = \frac{p_1 p_2}{4\pi\epsilon_0 r^3}(-\hat{\mathbf{x}})$$

(b) The force on dipole $\mathbf{p}$ due to the point charge is given by

$$\mathbf{F} = (\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E} = (\mathbf{p}\cdot\boldsymbol{\nabla})E_x\hat{\mathbf{x}} + (\mathbf{p}\cdot\boldsymbol{\nabla})E_y\hat{\mathbf{y}} + (\mathbf{p}\cdot\boldsymbol{\nabla})E_z\hat{\mathbf{z}}$$

The electric field due to the point charge q is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} = \frac{q}{4\pi\epsilon_0}\frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{3/2}}$$

So, we have

$$\begin{aligned}
F_x = (\mathbf{p}\cdot\boldsymbol{\nabla})E_x &= \left(p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} + p_z\frac{\partial}{\partial z}\right)\frac{q}{4\pi\epsilon_0}\frac{x}{(x^2+y^2+z^2)^{3/2}} \\
&= \frac{qp_x}{4\pi\epsilon_0}\left[\frac{1}{(x^2+y^2+z^2)^{3/2}} + \frac{(-3/2)x\times 2x}{(x^2+y^2+z^2)^{5/2}}\right] + \frac{qp_y}{4\pi\epsilon_0}\left[\frac{(-3/2)x\times 2y}{(x^2+y^2+z^2)^{5/2}}\right] + \frac{qp_z}{4\pi\epsilon_0}\left[\frac{(-3/2)x\times 2z}{(x^2+y^2+z^2)^{5/2}}\right] \\
&= \frac{q}{4\pi\epsilon_0}\left[\frac{p_x}{(x^2+y^2+z^2)^{3/2}} - \frac{3x(p_x x + p_y y + p_z z)}{(x^2+y^2+z^2)^{5/2}}\right] \\
&= \frac{q}{4\pi\epsilon_0}\left[\frac{p_x}{r^3} - \frac{3x(\mathbf{p}\cdot\mathbf{r})}{r^5}\right]
\end{aligned}$$

Therefore the force $\mathbf{F}$ is

$$\begin{aligned}
\mathbf{F} &= (\mathbf{p}\cdot\boldsymbol{\nabla})E_x\hat{\mathbf{x}} + (\mathbf{p}\cdot\boldsymbol{\nabla})E_y\hat{\mathbf{y}} + (\mathbf{p}\cdot\boldsymbol{\nabla})E_z\hat{\mathbf{z}} \\
&= \frac{q}{4\pi\epsilon_0}\left[\frac{p_x}{r^3} - \frac{3x(\mathbf{p}\cdot\mathbf{r})}{r^5}\right]\hat{\mathbf{x}} + \frac{q}{4\pi\epsilon_0}\left[\frac{p_y}{r^3} - \frac{3y(\mathbf{p}\cdot\mathbf{r})}{r^5}\right]\hat{\mathbf{y}} + \frac{q}{4\pi\epsilon_0}\left[\frac{p_z}{r^3} - \frac{3z(\mathbf{p}\cdot\mathbf{r})}{r^5}\right]\hat{\mathbf{z}} \\
&= \frac{q}{4\pi\epsilon_0 r^3}\left[\mathbf{p} - 3\hat{\mathbf{r}}(\mathbf{p}\cdot\hat{\mathbf{r}})\right]
\end{aligned}$$

Solution 5.2: Finding electric field in the presence of dielectric (Griffiths 3rd ed., Prob 4.15)

(a) Bound charges are calculated as follows:

$$\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) = -\frac{k}{r^2};$$
$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \frac{k}{r}\hat{\mathbf{r}}\cdot(\hat{\mathbf{r}}) = \frac{k}{b} \quad (\text{at } r=b)$$
$$= \frac{k}{r}\hat{\mathbf{r}}\cdot(-\hat{\mathbf{r}}) = -\frac{k}{a} \quad (\text{at } r=a)$$

(b) Because of the spherical symmetry, the Gauss's law implies $\mathbf{E} = \dfrac{Q_{\text{enc}}}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$. Therefore, we have

For $r<a$; $\quad Q_{\text{enc}} = 0$; $\quad$ so $\mathbf{E}=0$.
For $r>b$; $\quad Q_{\text{enc}} = 0$; $\quad$ so $\mathbf{E}=0$.
For $a<r<b$; $\quad Q_{\text{enc}} = \dfrac{-k}{a}4\pi a^2 + \displaystyle\int_a^r \frac{-k}{r'^2}4\pi r'^2 dr' = -4\pi ka - 4\pi k(r-a) = -4\pi kr;$ $\quad$ so $\quad \mathbf{E} = -\dfrac{k}{\epsilon_0 r}\hat{\mathbf{r}}$

(c) Since there are no free charges, the Gauss's law in the presence of dielectric gives

$$\oint \mathbf{D}\cdot d\mathbf{a} = 0 \;\Rightarrow \mathbf{D} = 0 \text{ at all } r$$

But we have $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ and so $\mathbf{E} = -\dfrac{\mathbf{P}}{\epsilon_0}$. Therefore we have

$$\text{For } r<a; \quad \mathbf{E} = -\frac{\mathbf{P}}{\epsilon_0} = 0.$$
$$\text{For } r>b; \quad \mathbf{E} = -\frac{\mathbf{P}}{\epsilon_0} = 0.$$
$$\text{For } a<r<b; \quad \mathbf{E} = -\frac{\mathbf{P}}{\epsilon_0} = -\frac{k}{\epsilon_0 r}\hat{\mathbf{r}}$$

Solution 5.3: Electric potential in a dielectric material (Griffiths 3rd ed., Prob 4.20)

In order to calculate the electric potential we need to calculate the electric field in the entire region. We'll use the Gauss's for electric displacement to calculate the displacement $\mathbf{D}(\mathbf{r})$ at $\mathbf{r}$ and from there calculate the electric field $\mathbf{E}(\mathbf{r})$ at $\mathbf{r}$.

$$\text{For } r<R; \qquad \oint \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}}$$
$$\text{Or,} \quad D4\pi r^2 = \rho\frac{4\pi}{3}r^3$$
$$\text{Or,} \quad D = \frac{\rho r}{3}$$
$$\Rightarrow \quad \mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \frac{\rho r}{3\epsilon}\hat{\mathbf{r}}$$

$$\text{For } r>R; \qquad \oint \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}}$$
$$\text{Or,} \quad D4\pi r^2 = \rho\frac{4\pi}{3}R^3$$
$$\text{Or,} \quad D = \frac{\rho R^3}{3r^2}$$
$$\Rightarrow \quad \mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \frac{\rho R^3}{3\epsilon_0 r^2}\hat{\mathbf{r}}$$

So, the electric potential V at the center of the sphere is

$$
\begin{aligned}
V &= -\int_{\infty}^{0} \mathbf{E} \cdot d\mathbf{l} \\
&= -\int_{\infty}^{R} \mathbf{E} \cdot d\mathbf{l} - \int_{R}^{0} \mathbf{E} \cdot d\mathbf{l} \\
&= -\int_{\infty}^{R} \frac{\rho R^3}{3\epsilon_0 r^2} dr - \int_{R}^{0} \frac{\rho r}{3\epsilon} dr \\
&= \frac{\rho R^2}{3\epsilon_0} + \frac{\rho}{3\epsilon}\frac{R^2}{2} \\
&= \frac{\rho R^2}{3\epsilon_0}\left(1 + \frac{1}{2\epsilon_r}\right)
\end{aligned}
$$

Solution 5.4: Capacitor with dielectric filling (Griffiths 3rd ed., Prob 4.21)

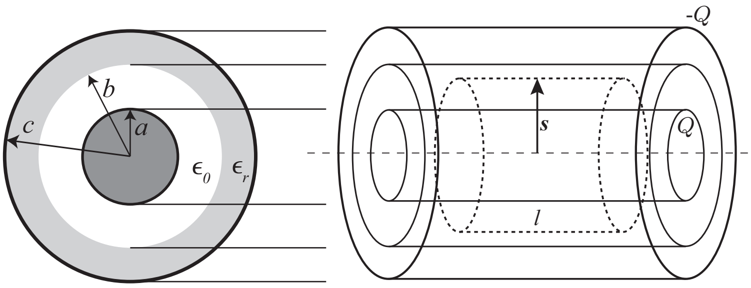

FIG. 1:

Let us assume that the inner cable has a charge Q for the length l of the cable. Therefore, using the Gauss's law for electric displacement we obtain

$$
\begin{aligned}
\text{For } a < s < b; \qquad & \oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}} \\
\text{Or,} \quad & D 2\pi s l = Q \\
\text{Or,} \quad & D = \frac{Q}{2\pi s l} \\
\Rightarrow \quad & \mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \frac{Q}{2\pi\epsilon_0 s l}\hat{\mathbf{s}}
\end{aligned}
$$

$$
\begin{aligned}
\text{For } b < r < c; \qquad & \oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}} \\
\text{Or,} \quad & D 2\pi s l = Q \\
\text{Or,} \quad & D = \frac{Q}{2\pi s l} \\
\Rightarrow \quad & \mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \frac{Q}{2\pi\epsilon s l}\hat{\mathbf{s}}
\end{aligned}
$$

So, the potential difference V is

$$
\begin{aligned}
V &= -\int_c^a \mathbf{E} \cdot d\mathbf{l} = \int_a^c \mathbf{E} \cdot d\mathbf{l} \\
&= \int_a^b \mathbf{E} \cdot d\mathbf{l} + \int_b^c \mathbf{E} \cdot d\mathbf{l} \\
&= \int_a^b \frac{Q}{2\pi\epsilon_0 sl} ds + \int_b^c \frac{Q}{2\pi\epsilon sl} ds \\
&= \frac{Q}{2\pi\epsilon_0 l} \left[\ln\left(\frac{b}{a}\right) + \frac{\epsilon_0}{\epsilon} \ln\left(\frac{c}{b}\right) \right]
\end{aligned}
$$

Therefore the capacitance per unit length is

$$
\frac{C}{l} = \frac{Q}{Vl} = \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right) + \frac{1}{\epsilon_r} \ln\left(\frac{c}{b}\right)}
$$

Solution 5.5: Force on a dielectric material (Griffiths 3rd ed., Prob 4.28)

We need to find the capacitance as a function of the height h and then use the formula for the force on a dielectric to obtain the final height that the oil would rise up to. In order to calculate the capacitance we'll need to calculate the potential and then the total charge. Assume that the total length of the tubes is l and that the oil rises up to height h. We assume that the line charge density of the air-part of the tubes is λ and that of the oil part is λ'. The potential can be calculated as follows:

$$
\begin{aligned}
\text{For the air part} \qquad & E_{\text{air}} = \frac{2\lambda}{4\pi\epsilon_0 s} \\
\Rightarrow \quad & V_{\text{air}} = \frac{2\lambda}{4\pi\epsilon_0} \ln\left(\frac{b}{a}\right) \\
\text{For the oil part} \qquad & D_{\text{oil}} = \frac{2\lambda'}{4\pi s} \\
\Rightarrow \quad & E_{\text{oil}} = \frac{2\lambda'}{4\pi\epsilon s} \\
\Rightarrow \quad & V_{\text{oil}} = \frac{2\lambda'}{4\pi\epsilon} \ln\left(\frac{b}{a}\right)
\end{aligned}
$$

We have the boundary condition that $V_{\text{oil}} = V_{\text{air}} = V$. Therefore we have $\lambda' = \frac{\epsilon}{\epsilon_0}\lambda = \epsilon_r \lambda$. The total charge on the tube is

$$
Q = \lambda' h + \lambda(l - h) = \epsilon_r \lambda h - \lambda h + \lambda l = \lambda[(\epsilon_r - 1) + l] = \lambda(\chi_e h + l)
$$

Therefore the capacitance of the system is

$$
C = \frac{Q}{V} = \frac{\lambda(\chi_e h + l)}{\frac{2\lambda}{4\pi\epsilon_0} \ln\left(\frac{b}{a}\right)} = 2\pi\epsilon_0 \frac{(\chi_e h + l)}{\ln\left(\frac{b}{a}\right)}
$$

Therefore, the force on the oil in the upward direction is

$$
F_1 = \frac{1}{2} V^2 \frac{dC}{dh} = \frac{1}{2} V^2 \frac{2\pi\epsilon_0 \chi_e}{\ln\left(\frac{b}{a}\right)}
$$

The gravitational force on the oil in the downward direction is

$$F_2 = mg = \rho\pi(b^2 - a^2)gh$$

The two forces become equal after the liquid rises up to height h. This is the maximum height of the liquid and is given by

$$h = \frac{\epsilon_0 \chi_e V^2}{\rho(b^2 - a^2)g\ln\left(\frac{b}{a}\right)}$$

Solution 5.6: bound charges in a cubical dielectric (Griffiths 3rd ed., Prob 4.31)

The volume charge density is given by $\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P} = -\boldsymbol{\nabla}\cdot(x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}) = -k(1+1+1) = -3k$. So the total volume charge is $Q_v = -3ka^3$. The surface charge density is given by $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$. For each surface, this quantity is $k\frac{a}{2}$. So, the total surface charge is $Q_s = 6 \times k\frac{a}{2} \times a^2 = 3ka^3$. So, the total bound charge is $Q = Q_v + Q_s = -3ka^3 + 3ka^3 = 0$, as expected.