PHY690G: Coherence and Quantum Entanglement **Semester I, 2018-19**

Homework # 1

Due on: Thursday, 16th August, 2018

Problem 1.1: (5+10=15 marks)

The mean $\langle x\rangle$ of a random process x is defined as $\langle x\rangle = \int xp(x)dx$, where $p(x)$ is the probability density associated with the random process. The standard deviation σ, which quantifies the degree of randomness of a process, is defined as $\sigma = \sqrt{\langle(x-\langle x\rangle)^2\rangle}$. Suppose $p_1(x)$ and $p_2(x)$ are the probability densities corresponding to two separate random processes and are defined as:

$$p_1(x) = \frac{1}{a} \quad \text{if} \quad 0 \le x \le a \quad \text{or else} \quad 0$$
$$p_2(x) = \frac{1}{b} \quad \text{if} \quad \frac{a-b}{2} \le x \le \frac{a+b}{2} \quad \text{or else} \quad 0 (\text{assume} \quad a > b)$$

(a) Verify that the above two probability densities are normalized.

(b) Calculate the mean and the standard deviation of both the random processes. Plot $p_1(x)$ and $p_2(x)$ as a function of x and show the corresponding mean and the standard deviation on the plots.

Problem 1.2: (5+10+10=25 marks)

One of the most often encountered random processes in Optics is the Gaussian random process. The probability density $p(x)$ associated with a Gaussian random process is given by

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \tag{1}$$

(a) Verify that the above distribution in normalized.

(b) Find the mean (first moment) and the second moment of this distribution. Also calculate the standard deviation.

(c) Calculate the third and the fourth moments. Do these moments yield any new information about the process?

Problem 1.3: (5+5+15+10+5=40 marks)

Consider the following random process:

$$Z(t) = \sum_{n=1}^{N} a_n e^{-i\omega_n t}, \tag{2}$$

in which $\omega_1, \omega_2, ...$ are fixed frequencies, but the coefficients $a_1, a_2, ...$ are random variables.

(a) First of all, explain what you understand by a stationary random process? What are the conditions for a process to be stationary at least in the wide sense?

(b) What are the conditions that ensures that $Z(t)$ is at least a wide-sense stationary process.

(c) Next, check whether or not the above process is ergodic. For that you have to check if the ergodicity condition $\lim_{\tau\to\infty} \left|\Gamma(\tau) - |\langle Z\rangle|^2\right| \to 0$ is satisfied. (Hint: use $\omega_n = \omega_0 + n\Delta\omega$).

(d) Assuming $|a_n|^2 = 1/N$, plot $\left|\Gamma(\tau) - |\langle Z\rangle|^2\right|$ as a function of τ for τ ranging from -1000 to 1000 and for (i) N=100 and $\Delta\omega = 0.1$ and (ii) N=100 and $\Delta\omega = 0.0001$. Comment on the main difference between these two cases?

(e) Comment on what happens to the ergodicity condition in situation in which $\Delta\omega \to 0$ and $N \to \infty$ in a way that $N\Delta\omega$ stays finite.

Problem 1.4: (15+5=20 marks) Let $V^{(r)}(\vec{r},t)$ be a real field and let's assume that it can be represented in terms of its Fourier components as:

$$\begin{aligned} V^{(r)}(\vec{r},t) &= \int_{-\infty}^{\infty} v(\vec{r},\omega)e^{-i\omega t}d\omega \\ &= \int_{-\infty}^{0} v(\vec{r},\omega)e^{-i\omega t}d\omega + \int_{0}^{\infty} v(\vec{r},\omega)e^{-i\omega t}d\omega = V^*(\vec{r},t) + V(\vec{r},t). \end{aligned}$$

$V(\vec{r},t)$ is called the complex analytic signal corresponding to the real variable $V^{(r)}(\vec{r},t)$. In coherence theory the time-averaged intensity $I(\vec{r})$ of a field at position $\vec{r}$ is defined as $I(\vec{r}) \propto \langle V^*(\vec{r},t)V(\vec{r},t)\rangle_t$. On the other hand, in classical electromagnetic theory, the intensity is given as $I^{(r)}(\vec{r}) = c\epsilon_0\langle V^{(r)2}(\vec{r},t)\rangle_t$. Show that:

(a)

$$\int_{-\infty}^{\infty} V^{(r)2}(\vec{r},t)dt = 2\int_{-\infty}^{\infty} V^*(\vec{r},t)V(\vec{r},t)dt = \int_{-\infty}^{\infty} |v(\vec{r},\omega)|^2 d\omega = 2\int_{0}^{\infty} |v(\vec{r},\omega)|^2 d\omega,$$

and thus prove that the two above mentioned definitions are related as

(b)

$$I^{(r)}(\vec{r}) = 2c\epsilon_0 I(\vec{r})$$