Homework # 2

Due in class on Thursday, August 30th, 2018

Problem 2.1: (2+3=5 marks) Suppose the cross-spectral density function of a field is given by $W(\omega_1, \omega_2) = S(\omega_1)\delta(\omega_1 - \omega_2)$, where $S(\omega) = \frac{1}{\sqrt{2\pi}\Delta\omega} \exp\left[-\frac{(\omega - \omega_0)^2}{2\Delta\omega^2}\right]$ with $\omega_0 \gg \Delta\omega$.

(a) Evaluate the intensity $I(t)$ of the field.

(b) Evaluate the coherence time of the field.

Problem 2.2: (4+2+2+2+2=12 marks) Consider a field $V(t)$ given by $V(t) = ae^{-i\omega_a t} + be^{-i\omega_b t}$, where ω_a and ω_b are two frequencies and a and b are two real numbers.

(a) Find out the cross-spectral density function $W(\omega_1, \omega_2)$ of the field.

(b) Find out the intensity $I(t)$ of the field.

(c) Find out the cross-correlation function $\Gamma(t_1, t_2)$ of the field.

(d) Is this field stationary, at least in the wide sense?

(e) Find out the degree of coherence function of the field.

Problem 2.3: (2+2+4+4+4+2=18 marks) Suppose the cross-spectral density function of a field is given by $W(\omega_1, \omega_2) = [S_+[\omega_1 - (\omega_0 + \Omega)] + S_-[\omega_1 - (\omega_0 - \Omega)]]\delta(\omega_1 - \omega_2)$, where $S_\pm[\omega - (\omega_0 \pm \Omega)] = \frac{1}{\sqrt{2\pi}\Delta\omega} \exp\left[-\frac{[\omega - (\omega_0 \pm \Omega)]^2}{2\Delta\omega^2}\right]$, with $\Delta\omega \ll \Omega \ll \omega_0$.

(a) Is this field wide-sense stationary in time?

(b) Plot the spectral density function $S(\omega)$ versus frequency ω. Indicate ω_0, Ω and $\Delta\omega$ on the plot.

(c) The above field enters a Mach-Zehnder interferomter with the time-delays in the two arms being t_1 and t_2. Calculate the intensity $I_0(\tau)$ at an output port of the interferometer in terms of $\tau = t_2 - t_1$. Assume $k_1 = k_2 = k$.

(d) Plot the output intensity $I_0(\tau)$ as a function of τ. Indicate ω_0, Ω and $\Delta\omega$ on the plot.

(e) Calculate the degree of coherence function and find out a reasonable estimate for the coherence time.

(f) Plot the degree of coherence as a function of τ and indicate the coherence time on the plot.

Problem 2.4: (10+5+10=25 marks)

For a spatially incoherent source the propagation equation within the far-field approximation takes the following form known as the van Cittert-Zernike theorem:

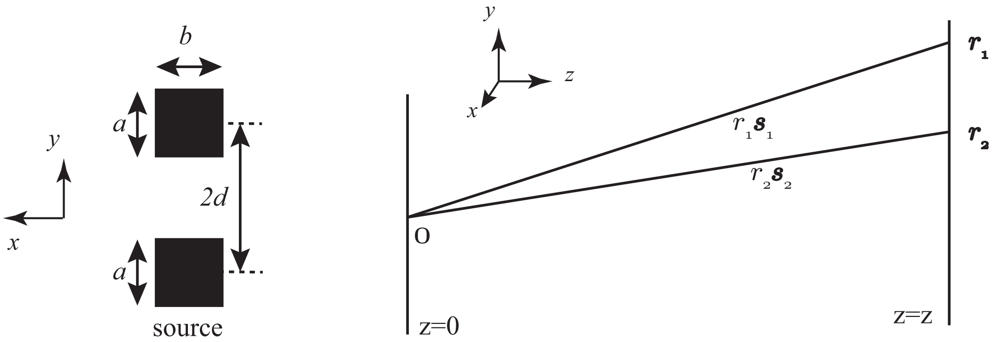
$$W(\mathbf{r_1}, \mathbf{r_2}, \omega) = -\left(\frac{k}{2\pi}\right)^2 \frac{\exp[ik(r_2 - r_1]}{r_1 r_2} \int_{\mathcal{A}} S(\mathbf{r}', \omega) e^{-ik(\mathbf{s_2} - \mathbf{s_1})\cdot\mathbf{r}'} d^2\mathbf{r}',$$

where $\mathbf{r}'$ is the source co-ordinate, $S(\mathbf{r}', \omega)$ is the spectral density of the sourece, $\mathbf{r_1} = r_1\mathbf{s_1}$ and $\mathbf{r_2} = r_2\mathbf{s_2}$ and where we have substituted $\Lambda_1^*(\bar{k})\Lambda_2(\bar{k}) = (-ik/2\pi)(ik/2\pi)$.

(a) Find the exact expression for the far-field expression for the cross-spectral density function if the source is spatially incoherent, has constant spectral density $S(\mathbf{r}', \omega) = S_0$ and is in the form of a square aperture of side a.

(b) Since we started with a spatially incoherent source, the coherence area at the source is of course zero. Make a reasonable estimate of the coherence area in the far field at z. Is the area finite? Comment on what caused the field to acquire a finite coherence area upon propagation.

(c) Consider the spatially incoherent source to be in the form as shown in the figure. The source is kept at $z = 0$. Again, assuming $S(\mathbf{r}', \omega) = S_0$ and $r_1 \approx r_2 \approx z$, derive the expression for the cross-spectral density $W(x_1, y_1, z; x_2, y_2, z; \omega)$ at $z = z$.



Problem 2.5: (5+5+10+5=25 marks)

(a) Consider the field $U(x, y, 0)$ of a Gaussian beam $U(x, y, 0) = A \exp\left[-\frac{x^2 + y^2}{w_0^2}\right]$ Where A is a constant. Is this field spatially coherent? What is the coherence area of such a field.

(b) What is the spectral amplitude $a(q_x, q_y)$ of such a beam. (hint: use the formula derived in the class).

(c) Next, consider the propagation of the Gaussian beam. First of all, show that the diffraction integral reduces to the following expression within the paraxial approximation: $z >> (x - x')^2 + (y - y')^2$.

$$U(x, y, z) = -\frac{ik}{2\pi z} e^{ikz} e^{ik(x^2+y^2)/2z} \iint U(x', y', 0) e^{ik(x'^2+y'^2)/2z} e^{-ik(xx'+yy')/z} dx'dy'$$

Next, evaluate $U(x, y, z)$ for the Gaussian beam with the amplitude $U(x', y', 0)$ given above, in terms of the following quantities: $z_R = \frac{kw_0^2}{2}$; $\quad w(z) = w_0\sqrt{1 + \frac{z^2}{z_R^2}}$; $\quad R(z) = z + \frac{z_R^2}{z}$.

(d) Is the beam at $z = z$ spatially coherent. What is the coherence area of the beam.

Problem 2.6: (15 marks) Consider a spatially-completely-coherent, constant-amplitude plane-wave field. The field is temporally stationary with the spectral density given by $S(\omega) = \frac{1}{\sqrt{2\pi}\Delta\omega} \exp\left[-\frac{(\omega - \omega_0)^2}{2\Delta\omega^2}\right]$ with $\omega_0 \gg \Delta\omega$. Suppose you are a given a Young's double slit setup. The separation between the two slits is $2d$ and the distance to the screen is R, with $R \gg d$, as shown in the figure. The size of the two slits are negligibly small. Describe how one could find the frequency bandwidth $\Delta\omega$ of the field by measuring the intensity $I(x)$ on the screen as a function of x.