Homework # 3

Due on: Thursday, September 6th, 2018

Problem 3.1: Wolf Equation (5+5=10 marks)

(a) In the class we derived the Wolf's Equation for the cross-correlation function to be

$$\nabla^2_{1(2)}\Gamma(\mathbf{r_1}, \mathbf{r_2}, \tau) = \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2}\Gamma(\mathbf{r_1}, \mathbf{r_2}, \tau)$$

Derive the Wolf's equation for the cross-spectral density function.

(b) In the class, we derived the propagation equation for the cross-spectral density function to be

$$W(\mathbf{r_1}, \mathbf{r_2}) = \int_{\mathcal{A}}\int_{\mathcal{A}} W(\mathbf{r'_1}, \mathbf{r'_2})\Lambda_1^*(k)\Lambda_2(k)\frac{\exp[ik(R_2 - R_1)]}{R_1 R_2} d^2\mathbf{r'_1} d^2\mathbf{r'_2} \tag{1}$$

Assume that $|R_2 - R_1|/c$ is much smaller compared to the coherence time of the source and that the cross-spectral density $W(\mathbf{r_1}, \mathbf{r_2}, \omega)$ is centered around the mean frequency $\bar{\omega} = c\bar{k}$ and has a quasi monochromatic spectrum. Derive the propagation equation for the cross-correlation function $\Gamma(\mathbf{r_1}, \mathbf{r_2}, \tau)$.

Problem 3.2: Propagation of Coherence (10+15=25 marks)

(a) The angular correlation function for the Gaussian Schell-model pump field is given by

$$\mathcal{A}(\boldsymbol{q_1}, \boldsymbol{q_2}) = A\exp\big[-\alpha(\boldsymbol{q_1})^2 - \alpha(\boldsymbol{q_2})^2 + 2\beta\boldsymbol{q_1}\cdot\boldsymbol{q_2}\big], \tag{2a}$$

where A is a constant. Show that the cross-spectral density function $W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$ at $z = 0$ is

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = A'\sqrt{S(\boldsymbol{\rho}_1)S(\boldsymbol{\rho}_2)}\mu(\Delta\boldsymbol{\rho}) = A'\sqrt{\exp\left[-\frac{(\boldsymbol{\rho}_1)^2}{2\sigma_s^2}\right]\exp\left[-\frac{(\boldsymbol{\rho}_2)^2}{2\sigma_s^2}\right]}\exp\left[-\frac{(\boldsymbol{\Delta}\rho)^2}{2\sigma_\mu^2}\right] \tag{3}$$

with $\alpha = \sigma_s^2\left(\sigma_\mu^2 + 2\sigma_s^2\right)/(\sigma_\mu^2 + 4\sigma_s^2)$ and $\beta = 2\sigma_s^4/(\sigma_\mu^2 + 4\sigma_s^2)$, $\boldsymbol{\Delta}\rho = \boldsymbol{\rho}_2 - \boldsymbol{\rho}_1$. Here σ_s is the size of beam at $z = 0$ and σ_μ is the transverse coherence width at $z = 0$, which is the distance scale over which the field at $z = 0$ remains spatially coherent.

(b) Next, calculate the far-field (at very large z) expression for the cross-spectral density function $W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z)$. Take the central wave-vector to be k_0.

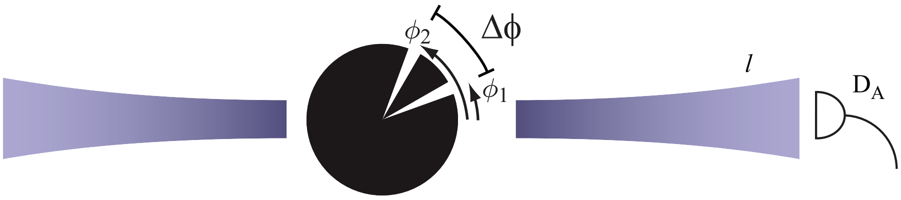
Problem 3.3: Constant Coherence Width (10 marks)

In the previous problem, we found that the transverse coherence length of the field depends on z. In this problem, find an angular correlation function of finite transverse spectral width $\boldsymbol{\Delta q}$ such that the transverse coherence length of the resultant field is finite but does not depend on the propagation distance z.

Problem 3.4: Angular interference (10+5=15 marks)

Consider the situation shown in the figure below. A stochastic field $V(\phi)$ falls onto an aperture in the form of a double-angular slit, with $V(\phi) = \sum_l \alpha_l e^{il\phi}$ and $\langle\alpha_{l_1}^*\alpha_{l_2}\rangle = C_{l_1}\delta_{l_1,l_2}$. Consider the angular slits to be infinitesimally narrow, that is, take the aperture function to be $\Phi(\phi) = \delta(\phi - \phi_1) + \delta(\phi - \phi_2)$. The detector D_A measures the intensity in a mode with orbital angular momentum (OAM) index l.

(a) Derive an expression for the intensity at the detector as a function of the OAM index l.



(b) Find out the explicit expression for the intensity at the detector as a function of the OAM index l, when $C_l = 1$, for $l = l_0$, and $C_l = 0$, otherwise.

Problem 3.5: Coherent-mode Representation (2+2+2+2+2=10 marks)

(a) What is the difference between a pure state and a mixed state?

(b) What is the difference between a coherent mode and an incoherent field. Can a coherent mode be a pure state?

(c) What is coherent-mode representation of a field and why does one need it? Is there any relation between the number of modes in the coherent-mode representation of a field and the degree of coherence?

(d) How many modes are there in the coherent-mode representation of a field that is completely coherent. Give an example of such a field.

(e) How many modes are there in the coherent-mode representation of a field that is completely incoherent. Give an example of such a field.

Problem 3.6: Unique decomposition of polarization matrix (10 marks)

Show that a general polarization matrix J can be uniquely expressed as a sum of two matrices, one of which is completely polarized and the other one completely unpolarized.

Problem 3.7: Polarization matrix and Stokes parameters (4+4+2=10 marks)

A polarization matrix can be expanded as a linear combination of identity matrix and three Pauli matrices: $J = \frac{1}{2}\sum_i S_i \sigma_i$, where S_i are called Stokes parameters and σ_i are the Pauli matrices, with $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\sigma_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$.

(a) First of all, show that for a general polarization density matrix $\det J \geq 0$.

(b) Show that $S_0^2 \geq S_1^2 + S_2^2 + S_3^2$

(c) Express the degree of polarization P, in terms of the Stokes parameters.

Problem 3.8: degree of polarization (8+1+1=10 marks)

Consider two separate and uncorrelated lasers. The field from the first laser is polarized along the $\hat{x}$ direction whereas the field from the other one is polarized along the $\hat{\theta}$ directions. The two laser fields are mixed at a beam splitter in equal proportion.

(a) What is the degree of polarization of the resultant field? **(8 marks)**

(b) What is the degree of polarization when $\theta = 0$? **(1 marks)**

(c) What is the degree of polarization when $\theta = \pi/2$? **(1 marks)**