Homework # 5

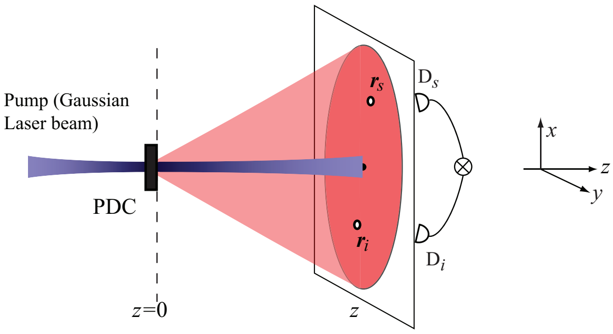
(Due in class on Monday, Nov 5th, 2018)

Problem 5.1: Misc Questions (5+5=10 marks)

(a) What interferes in a two-photon interference?

(b) What is orbital angular momentum of a photon described by the state $|\psi\rangle = \sum_{l=1}^{N} c_l |l\rangle$?

Problem 5.2: Spatial two-photon wave-function (5+20+5+10=40 marks)



Let us consider the production of entangled photons using parametric down-conversion as described in the figure. In class, we derived the expression for the two-photon state to be $|\psi_{\rm tp}\rangle = A \iint_{\infty}^{\infty} d^2\boldsymbol{q}_s d^2\boldsymbol{q}_i V(\boldsymbol{q}_s + \boldsymbol{q}_i)|\boldsymbol{q}_s\rangle|\boldsymbol{q}_i\rangle$.

(a) Within the assumptions of collinear down-conversion and paraxial approximation, show that the electric field operators at the two detectors are given by

$$\hat{E}_s^{(+)}(\boldsymbol{r}_s) = e^{ik_s z} \int d\boldsymbol{q} \hat{a}_s(\boldsymbol{q}) e^{i(\boldsymbol{q}\cdot\boldsymbol{\rho}_s - q^2 z/2k_s)}, \quad \text{and} \quad \hat{E}_i^{(+)}(\boldsymbol{r}_i) = e^{ik_i z} \int d\boldsymbol{q'} \hat{a}_i(\boldsymbol{q'}) e^{i(\boldsymbol{q'}\cdot\boldsymbol{\rho}_i - q'^2 z/2k_i)},$$

where $q^2 = |\boldsymbol{q}|^2$, ${q'}^2 = |\boldsymbol{q'}|^2$, $k_s = |\boldsymbol{k}_s(\omega_s)|$, and $k_i = |\boldsymbol{k}_i(\omega_i)|$.

(b) Next, take $k_s \approx k_i \approx k_0/2$, where k_0 is the cental wave-vector magnitude of the pump field, and consider the one-dimensional version of this problem so that $\boldsymbol{\rho} \to x$. Also, take the pump to be a completely spatially-coherent Gaussian beam: $V(k_{sx} + k_{ix}) = \exp\left[-w_0^2(k_{sx} + k_{ix})^2/4\right]$, where w_0 is the pump beam-waist. Show that the two-photon probability amplitude $\psi_{\rm tp}(x_s, x_i) = \langle {\rm vac}|\hat{E}_i^{(+)}(x_i)\hat{E}_s^{(+)}(x_s)|\psi_{\rm tp}\rangle$ is given by

$$\psi_{\rm tp}(x_s, x_i) = A e^{ik_0 z} \exp\left[-\frac{(x_s + x_i)^2}{4w^2(z)}\right] \exp\left[\frac{i(x_s + x_i)^2 z}{2k_0 w_0^2 w^2(z)}\right] \exp\left[\frac{ik_0(x_s - x_i)^2}{8z}\right],$$

where $w(z) = [w_0^2 + (4z^2)/(k_0^2 w_0^2)]^{1/2}$ is the beam-waist of the pump at $z = z$, and A is a z-dependent constant.

(c) Finally, show that the two-photon coincidence probability, to within an overall z-dependent constant, is

$$R(x_s, x_i) = |\psi_{\rm tp}(x_s, x_i)|^2 = \langle\psi_{\rm tp}|\hat{E}_s^{(-)}(x_s)\hat{E}_i^{(-)}(x_i)\hat{E}_i^{(+)}(x_i)\hat{E}_s^{(+)}(x_s)|\psi_{\rm tp}\rangle \to A \exp\left[-\frac{(x_s + x_i)^2}{2w^2(z)}\right]$$

(d) Plot the above two-photon coincidence probability $R(x_s, x_i)$ and describe it.

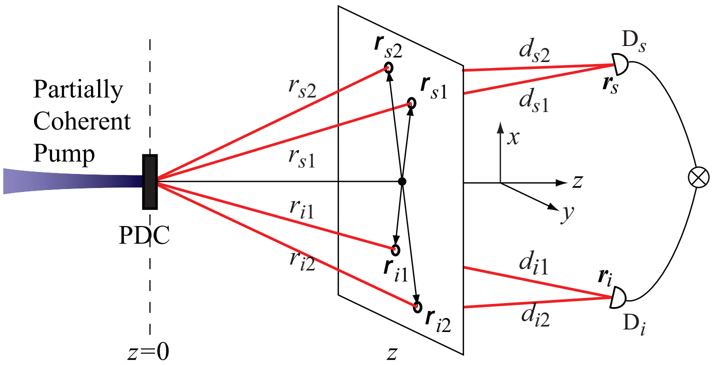
Problem 5.3: Spatial Coherence and Entanglement (5+5+5+5+5+5=30 marks)

The scheme below represents a spatial two-qubit state, with $\{|s1\rangle, |s2\rangle\}$ and $\{|i1\rangle, |i2\rangle\}$ forming the two-dimensional orthonormal bases for the signal and idler photons, respectively, where $|s1\rangle$ represents the state of the signal photon passing through the hole located at transverse position $\boldsymbol{\rho}_{s1}$, etc. The four-dimensional basis-set for the two-qubit state can then be represented by $\{|s1\rangle|i1\rangle, |s1\rangle|i2\rangle, |s2\rangle|i1\rangle, |s2\rangle|i2\rangle\}$, where $|s1\rangle|i1\rangle$ represents the joint state of the signal and idler photons when the signal photon passes through the hole located at $\boldsymbol{\rho}_{s1}$ and the idler photon passes through the hole located at $\boldsymbol{\rho}_{i1}$, etc. Assuming that the probabilities of states $|s1\rangle|i2\rangle$ and $|s2\rangle|i1\rangle$ are negligibly small, we write the density matrix ρ_{qubit} of the two-qubit state thus prepared as:

$$\rho_{\text{qubit}} = \begin{pmatrix} a & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d & 0 & 0 & b \end{pmatrix}. \tag{1}$$

The coincidence count rate at the detector, as derived in the class is

$$R_{si}(\boldsymbol{r}_s, \boldsymbol{r}_i) = k_1^2 S^{(2)}(\boldsymbol{\rho}_1, z) + k_2^2 S^{(2)}(\boldsymbol{\rho}_2, z) + 2k_1 k_2 \sqrt{S^{(2)}(\boldsymbol{\rho}_1, z) S^{(2)}(\boldsymbol{\rho}_2, z)} |\mu^{(2)}(\Delta\boldsymbol{\rho}, z)| \cos(k_0 \Delta L + \Delta\phi).$$



(a) What is the meaning of the diagonal terms of the density matrix ρ_{qubit}? Explain why a and b should be proportional to the two-photon spectral densities, that is, $a = k_1^2 S^{(2)}(\boldsymbol{\rho}_1, z)$ and $b = k_2^2 S^{(2)}(\boldsymbol{\rho}_2, z)$?

(b) What is the meaning of the off-diagonal terms of the density matrix ρ_{qubit}? Explain why $c = d^*$ should be proportional to the two-photon cross-spectral density, that is, $c = d^* = k_1 k_2 W^{(2)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z)$?

(c) Calculate the concurrence $C(\rho_{\text{qubit}})$ of the above two-qubit state.

(d) Find the visibility V of two-photon fringes and show that $V = C(\rho_{\text{qubit}})$.

(e) What is $C(\rho_{\text{qubit}})$ when $a = b$ and $|c| = |d| = 1/2$? What is the necessary physical condition for ensuring this?

(f) What is $C(\rho_{\text{qubit}})$ when $a = b$ and $|c| = |d| = 0$? What is the necessary physical condition for ensuring this?

Problem 5.4: Concurrence of an X-matrix (2+6+2+10=20 marks)

Consider a two-particle state $|\psi_1\rangle = a|HH\rangle + b|VV\rangle$, where a and b are real numbers and where $a^2 + b^2 = 1$. Consider another two-particle state $|\psi_2\rangle = p|HV\rangle + q|VH\rangle$, where p and q are real numbers and where $p^2 + q^2 = 1$.

(a) Write down the density matrix ρ_1 corresponding to state $|\psi_1\rangle$.

(b) Calculate the concurrence of ρ_1.

(c) Now, consider a two-particle state that is an incoherent mixture of staes $|\psi_1\rangle$ and $|\psi_2\rangle$ with equal proportion. Write down the density matrix ρ corresponding to this state.

(d) Calculate the concurrence of ρ.