Solutions: Homework # 1

Solution 1.1

(a)

$$\int_0^a \frac{1}{a}dx = 1$$
$$\int_{(a-b)/2}^{(a+b)/2} \frac{1}{b}dx = \frac{1}{b}\left[\frac{a+b}{2} - \frac{a-b}{2}\right] = 1$$

So both the probabilities are normalized.

(b)

$$P_1(x) = \frac{1}{a}$$
$$\text{mean} : \langle x\rangle = \int_0^a x p_1(x)dx = \frac{1}{a}\int_0^a x dx = \frac{a}{2}$$
$$\text{second moment} : \langle x^2\rangle = \int_0^a x^2 p_1(x)dx = \frac{1}{a}\int_0^a x^2 dx = \frac{1}{a}\left[\frac{x^3}{3}\right]_0^a = \frac{a^2}{3}$$
$$\text{Standard Deviation} : \sigma = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{a^2}{3} - \frac{a^2}{4}} = a\sqrt{\frac{4-3}{12}} = \frac{a}{2\sqrt{3}}$$

$$P_2(x) = \frac{1}{b}$$
$$\text{mean} : \langle x\rangle = \int_{(a-b)/2}^{(a+b)/2} x p_2(x)dx = \frac{1}{b}\int_{(a-b)/2}^{(a+b)/2} x dx = \frac{a}{2}$$
$$\text{second moment} : \langle x^2\rangle = \int_{(a-b)/2}^{(a+b)/2} x^2 p_2(x)dx = \frac{1}{b}\int_{(a-b)/2}^{(a+b)/2} x^2 dx = \frac{1}{b}\left[\frac{x^3}{3}\right]_{(a-b)/2}^{(a+b)/2}$$
$$= \frac{1}{3b}\left[\frac{(a+b)^3 - (a-b)^3}{8}\right] = \frac{3a^2+b^2}{12}$$
$$\text{Standard Deviation} : \sigma = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{3a^2+b^2}{12} - \frac{a^2}{4}} = \frac{b}{2\sqrt{3}}$$

Solution 1.2

(a) Before calculating the means, we'll first solve the Gaussian integral:

$$I = \int_{-\infty}^{\infty} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx$$

Substitute $x - \mu = x'$. We get,

$$I = \int_{-\infty}^{\infty} \exp\left[-\frac{x'^2}{2\sigma^2}\right] dx' = \int_{-\infty}^{\infty} \exp\left[-\frac{x^2}{2\sigma^2}\right] dx$$

And therefore,

$$I^2 = I \times I = \int_{-\infty}^{\infty} \exp\left[-\frac{x^2}{2\sigma^2}\right] dx \times \int_{-\infty}^{\infty} \exp\left[-\frac{y^2}{2\sigma^2}\right] dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-\frac{(x^2+y^2)}{2\sigma^2}\right] dxdy$$

Switch to polar coordinate: $x = r\cos\theta$, $y = r\sin\theta$, and

$$\begin{aligned} dxdy &= |J| drd\theta = \left|\frac{\partial(x,y)}{\partial(r,\theta})\right| drd\theta = rdrd\theta \\ \text{where} \quad |J| &= \left|\frac{\partial(x,y)}{\partial(r,\theta})\right| = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r \end{aligned}$$

Thus

$$I^2 = \int_0^{2\pi}\int_0^{\infty} \exp\left[-\frac{r^2}{2\sigma^2}\right] rdrd\theta$$

Now substitute $r^2 = z$ so that $dz = 2rdr$ and

$$\begin{aligned} I^2 &= \pi \int_0^{\infty} \exp\left[-\frac{z}{2\sigma^2}\right] dz = \pi \left[\frac{\exp[-z^2/(2\sigma^2)]}{-1/(2\sigma^2)}\right]_0^{\infty} = 2\pi\sigma^2 \\ \Rightarrow I &= \int_{-\infty}^{\infty} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx = \sqrt{2\pi}\sigma \end{aligned} \tag{1}$$

Check the normalization:

$$\int_{-\infty}^{\infty} p(x)dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx = \frac{1}{\sqrt{2\pi}\sigma}.\sqrt{2\pi}\sigma = 1.$$

(b)

$$\text{mean} : \langle x \rangle = \int_{-\infty}^{\infty} xp(x)dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx.$$

Substituting $x - \mu = x'$ and using the result in Eq. (1), we get

$$\begin{aligned} \langle x \rangle &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x'+\mu)\exp\left[-\frac{x'^2}{2\sigma^2}\right] dx' \\ &= \frac{1}{\sqrt{2\pi}\sigma}\left[\int_{-\infty}^{\infty} x'\exp\left[-\frac{x'^2}{2\sigma^2}\right] dx' + \mu\int_{-\infty}^{\infty} \exp\left[-\frac{x'^2}{2\sigma^2}\right] dx'\right] \\ &= \frac{1}{\sqrt{2\pi}\sigma}\left[0 \quad (\text{odd function}) + \mu\sqrt{2\pi}\sigma\right] = \mu. \end{aligned}$$

$$\text{second moment} : \langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 p(x)dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x^2\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx.$$

Substituting $x - \mu = x'$, we get

$$\begin{aligned} \langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 p(x)dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x'+\mu)^2\exp\left[-\frac{x'^2}{2\sigma^2}\right] dx' \\ &= \frac{1}{\sqrt{2\pi}\sigma}\left[\int_{-\infty}^{\infty} x'^2\exp\left[-\frac{x'^2}{2\sigma^2}\right] dx' + \mu'^2\int_{-\infty}^{\infty} \exp\left[-\frac{x'^2}{2\sigma^2}\right] dx' + 2\mu\int_{-\infty}^{\infty} x'\exp\left[-\frac{x'^2}{2\sigma^2}\right] dx'\right] \\ &= \frac{1}{\sqrt{2\pi}\sigma}\left[I_2 + \mu^2\sqrt{2\pi}\sigma + 0\right] \end{aligned} \tag{2}$$

We now need to calculate I_2. Substituting $\alpha = 1/(2\sigma^2)$ and using Eq. (1), we get

$$I_2 \equiv \int_{-\infty}^{\infty} x'^2 \exp\left[-\frac{x'^2}{2\sigma^2}\right] dx' = \int_{-\infty}^{\infty} x'^2 \exp\left[-\alpha x'^2\right] dx' = -\frac{d}{d\alpha}\int_{-\infty}^{\infty} \exp\left[-\alpha x'^2\right] dx'$$

$$= -\frac{d}{d\alpha}\left[\sqrt{2\pi} \times \sqrt{\frac{1}{2\alpha}}\right] = \frac{1}{2\alpha}\left(\frac{\pi}{\alpha}\right)^{1/2} = \sigma^2.[2\pi\sigma^2]^{1/2} = \sqrt{2\pi}\sigma^3$$

Substituting for I_2 in Eq. (2) gives

$$\langle x^2 \rangle = \frac{1}{\sqrt{2\pi}\sigma}\left[\sqrt{2\pi}\sigma^3 + \sigma\mu^2\sqrt{2\pi}\right] = [\sigma^2 + \mu^2]$$

$$\text{Standard Deviation} = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\sigma^2 + \mu^2 - \mu^2} = \sigma$$

(c) Now, using Eq. (1) and repeating the trick used for calculating I_2, we can show that

$$\langle x^3 \rangle = \mu^3 + 3\mu\sigma^2$$
$$\langle x^4 \rangle = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$$

Solution 1.3

(a) If all the probabilities governing the random process are invariant under an arbitrary translation of the origin of time, then that process is referred to as the stationary random process. For a process to be strictly stationary, all the probabilities need to be invariant. However, if at least the mean and the auto-correlation function are invariant under an arbitrary translation of the origin of time, then the process is referred to as a wide-sense stationary process.

(b)

$$Z(t) = \sum_{n=1}^{N} a_n e^{-i\omega_n t}, \tag{3}$$

If $Z(t)$ is to be a wide-sense stationary process, the mean should be independent of time and the auto-correlation function should depend only on the time difference. The mean and the auto-correlation function are defined as

$$\text{mean} : \langle Z(t)\rangle = \sum_{n=1}^{N} \langle a_n\rangle e^{-i\omega_n t},$$

$$\text{auto-correlation function} : \Gamma(t, t+\tau) = \langle Z^*(t)Z(t+\tau)\rangle = \sum_{n=1}^{N}\sum_{m=1}^{N} \langle a_n^* a_m\rangle e^{i(\omega_n - \omega_m)t} e^{-i\omega_m \tau}$$

Therefore to ensure that the process is at least a wide-sense stationary random process, the following two conditions would need to be satisfied:

$$\langle a_n \rangle = 0 \quad \forall n,$$
$$\langle a_n^* a_m \rangle = \langle |a_n|^2\rangle \delta_{nm}$$

(c) We have

$$\Gamma(\tau) = \sum_{n=1}^{N} |a_n|^2 e^{-i\omega_n \tau}$$

Let's first center the distribution around the center frequency ω_0 such that

$$\Gamma(\tau) = \sum_{m=-N/2}^{N/2} |a_m|^2 e^{-i(\omega_0 + m\Delta\omega)\tau} = e^{-i\omega_0\tau} \sum_{m=-N/2}^{N/2} |a_m|^2 e^{-im\Delta\omega\tau}$$

Next substitute $|a_n|^2 = \text{Rect}(m/N) \times g_m$, where $\text{Rect}(m/N) = 1$, when $-N/2 < m < N/2$ or else 0. Thus we can now write

$$\Gamma(\tau) = e^{-i\omega_0\tau} \sum_{m=-\infty}^{\infty} \text{Rect}\left(\frac{m}{N}\right) g_m e^{-im\Delta\omega\tau}$$

Since here m are integers, we do one more trick to convert the summation into an integral, i.e.,

$$\begin{aligned}\Gamma(\tau) =& e^{-i\omega_0\tau} \int_{-\infty}^{\infty} \text{Rect}\left(\frac{m}{N}\right) g(m)\text{Comb}(m) e^{-im\Delta\omega\tau} dm \\ =& e^{-i\omega_0\tau} \int_{-\infty}^{\infty} h(m)\text{Comb}(m) e^{-im\Delta\omega\tau} dm\end{aligned}$$

where $\text{Rect}\left(\frac{m}{N}\right) g(m) = h(m)$ and $\text{Comb}(m) = \sum_{l=-\infty}^{\infty} \delta(m-l)$. We note that the integral is the Fourier transform with $\Delta\omega\tau$ being the transform variable. Therefore we can write.

$$\Gamma(\tau) = \sqrt{2\pi} e^{-i\omega_0\tau} \mathcal{F}[h(m)\text{Comb}(m)] = \sqrt{2\pi} e^{-i\omega_0\tau} \mathcal{F}[h(m)] \otimes \mathcal{F}[\text{Comb}(m)]$$

where $\mathcal{F}$ represents Fourier transformation and $\otimes$ represents convolution. After taking the Fourier transforms we can write the above equation as

$$\Gamma(\tau) = \sqrt{2\pi} e^{-i\omega_0\tau} H(\Delta\omega\tau) \otimes \frac{1}{\sqrt{2\pi}} \text{Comb}\left(\frac{\Delta\omega\tau}{2\pi}\right) = e^{-i\omega_0\tau} H(\Delta\omega\tau) \otimes \text{Comb}\left(\frac{\Delta\omega\tau}{2\pi}\right)$$

Here $H(\Delta\omega\tau) = \mathcal{F}[h(m)]$. Taking the convolution and using the definition of the Comb function we, get

$$\begin{aligned}\Gamma(\tau) =& e^{-i\omega_0\tau} \int_{-\infty}^{\infty} \text{Comb}\left(\frac{x}{2\pi}\right) H(x - \Delta\omega\tau) dx = e^{-i\omega_0\tau} \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta\left(\frac{x}{2\pi} - k\right) H(x - \Delta\omega\tau) dx \\ =& e^{-i\omega_0\tau} \sum_{k=-\infty}^{\infty} H(2\pi k - \Delta\omega\tau)\end{aligned}$$

Similarly it can be shown that

$$\Gamma^*(\tau) = e^{i\omega_0\tau} \sum_{k=-\infty}^{\infty} H(2\pi k + \Delta\omega\tau)$$

Therefore

$$|\Gamma(\tau)|^2 = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} H(2\pi k + \Delta\omega\tau) H(2\pi l - \Delta\omega\tau)$$

We know that $H(\Delta\omega\tau)$ is the Fourier transform of a function that has a finite width in the frequency space. Thus we expect $H(\Delta\omega\tau)$ to be a function in the τ domain with a finite width. The above sum is basically an infinite series of displaced $H(\Delta\omega\tau)$ function. Thus it is clear that $|\Gamma(\tau)|$ does not tend to zero in the limit when $\tau \to \infty$.

(d) Please note that since in this case $g(n)$=constant, $H(\Delta\omega\tau)$ is the Fourier transform of a Rect function which is a Sinc function. The main difference between the two cases is that τ-separation between two consecutive displaced $H(\Delta\omega\tau)$ function is much larger for smaller $\Delta\omega$ since the separation depends on $1/\Delta\omega$.

(e) When $\Delta\omega$ goes to zero, the τ-separation between the two displaced $H(\Delta\omega\tau)$ functions is infinite. And therefore one would never encounter the second displaced $H(\Delta\omega\tau)$ function before infinity. So, in this sense we can say that $|\Gamma(\tau)|$ has now become ergodic. This can also be seen if we rewrite the $|\Gamma(\tau)|$ in the limit when $\Delta\omega$ goes to zero. $|\Gamma(\tau)|$ now becomes.

$$\Gamma(\tau) = e^{-i\omega_0\tau}\int_{-\infty}^{\infty}|a(\omega)|^2 e^{-i\omega\tau}d\omega$$

Please note that we have assumed here that $|a(\omega)|$ is a well-behaved function of frequency with a finite frequency-width (quasi-monochromaticity).

Solution 1.4

(a) Using the definition for $V^{(r)}(\vec{r},t)$, we get

$$\begin{aligned}\int_{-\infty}^{\infty}V^{(r)2}(\vec{r},t)dt =& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}v(\vec{r},\omega)v(\vec{r},\omega')e^{-i(\omega+\omega')t}d\omega d\omega' dt \\ =&2\pi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}v(\vec{r},\omega)v(\vec{r},\omega')\delta(\omega+\omega')d\omega d\omega' = 2\pi\int_{-\infty}^{\infty}v(\vec{r},\omega)v(\vec{r},-\omega)d\omega \\ =&2\pi\int_{-\infty}^{\infty}v(\vec{r},\omega)v^*(\vec{r},\omega)d\omega = 2\pi\int_{-\infty}^{\infty}|v(\vec{r},\omega)|^2d\omega = 2\times 2\pi\int_{0}^{\infty}|v(\vec{r},\omega)|^2d\omega\end{aligned} \tag{4}$$

Next, using the definition for $V(\vec{r},t)$, we show that

$$\begin{aligned}\int_{-\infty}^{\infty}V^*(\vec{r},t)V(\vec{r},t)dt =& \int_{-\infty}^{\infty}\left[\int_{-\infty}^{0}v(\vec{r},\omega)e^{-i\omega t}d\omega\int_{0}^{\infty}v(\vec{r},\omega')e^{-i\omega' t}d\omega'\right]dt \\ & \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}v^*(\vec{r},\omega)v(\vec{r},\omega')e^{i(\omega-\omega')t}d\omega d\omega' dt \\ =&2\pi\int_{0}^{\infty}\int_{0}^{\infty}v^*(\vec{r},\omega)v(\vec{r},\omega')\delta(\omega-\omega')d\omega d\omega' = 2\pi\int_{0}^{\infty}v^*(\vec{r},\omega)v(\vec{r},\omega)d\omega \\ =&2\pi\int_{0}^{\infty}|v(\vec{r},\omega)|^2d\omega\end{aligned} \tag{5}$$

We also have that

$$\begin{aligned}\int_{-\infty}^{\infty}|v(\vec{r},\omega)|^2d\omega =& \int_{-\infty}^{0}|v(\vec{r},\omega)|^2d\omega + \int_{0}^{\infty}|v(\vec{r},\omega)|^2d\omega = \int_{-\infty}^{0}v^*(\vec{r},\omega)v(\vec{r},\omega)d\omega + \int_{0}^{\infty}|v(\vec{r},\omega)|^2d\omega \\ =& \int_{-\infty}^{0}v(\vec{r},-\omega)v^*(\vec{r},-\omega)d\omega + \int_{0}^{\infty}|v(\vec{r},\omega)|^2d\omega \\ =& \int_{0}^{\infty}v(\vec{r},\omega)v^*(\vec{r},\omega)d\omega + \int_{0}^{\infty}|v(\vec{r},\omega)|^2d\omega = 2\int_{0}^{\infty}|v(\vec{r},\omega)|^2d\omega\end{aligned} \tag{6}$$

Using Eqs. (4), (5) and (6), we get the desired result.

$$\int_{-\infty}^{\infty}V^{(r)2}(\vec{r},t)dt = 2\int_{-\infty}^{\infty}V^*(\vec{r},t)V(\vec{r},t)dt = 2\pi\int_{-\infty}^{\infty}|v(\vec{r},\omega)|^2d\omega = 2\times 2\pi\int_{0}^{\infty}|v(\vec{r},\omega)|^2d\omega,$$

(b) Again, using Eqs. (4), (5) and (6), we find that

$$I^{(r)}(\vec{r}) = c\epsilon_0\langle V^{(r)2}(\vec{r},t)\rangle_t = 2c\epsilon_0\langle V^2(\vec{r},t)\rangle_t$$