Solutions: Homework # 3

Solution 3.1

(a) We start with the Wolf equation for the cross-correlation function:

$$\nabla_1^2 \Gamma(\mathbf{r_1}, \mathbf{r_2}, \tau) = \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2}\Gamma(\mathbf{r_1}, \mathbf{r_2}, \tau)$$

Writing the cross-correlation function as the Fourier transform of the cross-spectral density function and simplifying, we get

$$\begin{aligned}
&\nabla_1^2 \int_0^\infty W(\mathbf{r_1}, \mathbf{r_2}, \omega) e^{-i\omega\tau} d\omega = \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2}\int_0^\infty W(\mathbf{r_1}, \mathbf{r_2}, \omega) e^{-i\omega\tau} d\omega \\
\text{or,}\quad &\nabla_1^2 \int_0^\infty W(\mathbf{r_1}, \mathbf{r_2}, \omega) e^{-i\omega\tau} d\omega = \frac{1}{c^2}\int_0^\infty W(\mathbf{r_1}, \mathbf{r_2}, \omega) \frac{\partial^2}{\partial \tau^2} e^{-i\omega\tau} d\omega \\
\text{or,}\quad &\nabla_1^2 \int_0^\infty W(\mathbf{r_1}, \mathbf{r_2}, \omega) e^{-i\omega\tau} d\omega = \frac{-\omega^2}{c^2}\int_0^\infty W(\mathbf{r_1}, \mathbf{r_2}, \omega) e^{-i\omega\tau} d\omega \\
\text{or,}\quad &\int_0^\infty \left(\nabla_1^2 W(\mathbf{r_1}, \mathbf{r_2}, \omega) + \frac{\omega^2}{c^2} W(\mathbf{r_1}, \mathbf{r_2}, \omega)\right) e^{-i\omega\tau} d\omega = 0 \\
\Rightarrow\quad &\nabla_1^2 W(\mathbf{r_1}, \mathbf{r_2}, \omega) + |\mathbf{k}|^2 W(\mathbf{r_1}, \mathbf{r_2}, \omega) = 0
\end{aligned}$$

In a similar manner, one could derive the Wolf's equation for the other space derivative.

(b) The propagation equation for the cross-spectral density function is:

$$W(\mathbf{r_1}, \mathbf{r_2}, \omega) = \int_{\mathcal{A}}\int_{\mathcal{A}} W(\mathbf{r'_1}, \mathbf{r'_2}, \omega)\Lambda_1^*(k)\Lambda_2(k)\frac{\exp[ik(R_2 - R_1)]}{R_1 R_2} d^2\mathbf{r'_1} d^2\mathbf{r'_2} \tag{1}$$

Taking the Fourier transform of this equation on both sides we get,

$$\int_0^\infty W(\mathbf{r_1}, \mathbf{r_2}, \omega) e^{-i\omega\tau} d\omega = \int_0^\infty \left(\int_{\mathcal{A}}\int_{\mathcal{A}} W(\mathbf{r'_1}, \mathbf{r'_2}, \omega)\frac{\exp[ik(R_2 - R_1)]}{R_1 R_2}\Lambda_1^*(k)\Lambda_2(k) d^2\mathbf{r'_1} d^2\mathbf{r'_2}\right) e^{-i\omega\tau} d\omega$$

Writing, $k = \omega/c$, we write the above equation as

$$\int_0^\infty W(\mathbf{r_1}, \mathbf{r_2}, \omega) e^{-i\omega\tau} d\omega = \int_0^\infty \left(\int_{\mathcal{A}}\int_{\mathcal{A}} W(\mathbf{r'_1}, \mathbf{r'_2}, \omega)\frac{\exp[i\omega(R_2 - R_1)/c]}{R_1 R_2}\Lambda_1^*(\omega/c)\Lambda_2(\omega/c) d^2\mathbf{r'_1} d^2\mathbf{r'_2}\right) e^{-i\omega\tau} d\omega$$

We work in the quasi-monochromatic limit, we assume that the inclination factors $\Lambda_1^*(\omega/c)$ and $\Lambda_2^*(\omega/c)$ do not change appreciably over the frequency range of interest, so they can be pulled out of the ω-integral after replacing them with their values at mean frequencies $\Lambda_1^*(\bar{\omega}/c)$ and $\Lambda_2^*(\bar{\omega}/c)$, respectively. After rearranging, we can now write

$$\int_0^\infty W(\mathbf{r_1}, \mathbf{r_2}, \omega) e^{-i\omega\tau} d\omega = \int_{\mathcal{A}}\int_{\mathcal{A}}\int_0^\infty \left(W(\mathbf{r'_1}, \mathbf{r'_2}, \omega)\frac{\exp[i\omega(R_2 - R_1)/c]}{R_1 R_2} e^{-i\omega\tau} d\omega\right)\Lambda_1^*(\bar{\omega}/c)\Lambda_2(\bar{\omega}/c) d^2\mathbf{r'_1} d^2\mathbf{r'_2}$$

Now let's consider the exponential:

$$\exp\left[i\omega\frac{(R_2-R_1)}{c}\right] = \exp\left[i\bar{\omega}\frac{(R_2-R_1)}{c}\right]\exp\left[i(\omega-\bar{\omega})\frac{(R_2-R_1)}{c}\right]$$

We have that $|R_2 - R_1|/c$ is much smaller than the coherence time of the source, that is, that $|R_2 - R_1|/c \ll 1/\Delta\omega$, where $\Delta\omega$ is the frequency range of interest. Also, for the frequency range of interest $\omega - \bar{\omega} \leq \Delta\omega$. Therefore, within the frequency range of interest $(\omega - \bar{\omega})\frac{(R_2-R_1)}{c} \ll 1$. With this approximation, we have

$$\exp\left[i\bar{\omega}\frac{(R_2-R_1)}{c}\right]\exp\left[i(\omega-\bar{\omega})\frac{(R_2-R_1)}{c}\right] \approx \exp\left[i\bar{\omega}\frac{(R_2-R_1)}{c}\right]$$

Substituting in the above equation we get

$$\int_0^\infty W(\mathbf{r_1},\mathbf{r_2},\omega)e^{-i\omega\tau}d\tau$$
$$= \int_{\mathcal{A}}\int_{\mathcal{A}}\left(\int_0^\infty W(\mathbf{r'_1},\mathbf{r'_2},\omega)e^{-i\omega\tau}d\tau\right)\frac{\exp[i\bar{\omega}(R_2-R_1)/c]}{R_1R_2}\Lambda_1^*(\bar{\omega}/c)\Lambda_2(\bar{\omega}/c)d^2\mathbf{r'_1}d^2\mathbf{r'_2}$$

Now, using the definition of the cross-correlation function, and replacing $\bar{\omega} = c\bar{k}$, we get

$$\Gamma(\mathbf{r_1},\mathbf{r_2},\tau) = \int_{\mathcal{A}}\int_{\mathcal{A}}\Gamma(\mathbf{r'_1},\mathbf{r'_2},\tau)\frac{\exp[i\bar{k}(R_2-R_1)]}{R_1R_2}\Lambda_1^*(\bar{k})\Lambda_2(\bar{k})d^2\mathbf{r'_1}d^2\mathbf{r'_2}, \tag{2}$$

which is the propagation equation of the cross-correlation function.

Solution 3.2:

(a) See Mandel and Wolf, Section 5.6.4 for the detailed derivation. The cross-spectral density at $z = 0$ is given by

$$W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = \frac{1}{k^4}\iint_{-\infty}^{\infty}\mathcal{A}(\boldsymbol{q_1},\boldsymbol{q_2})e^{-i(\boldsymbol{q_1}\cdot\boldsymbol{\rho}_1-\boldsymbol{q_2}\cdot\boldsymbol{\rho}_2)}d^2\boldsymbol{q_1}d^2\boldsymbol{q_2}.$$

Substituting the expression for the angular coherence function $\mathcal{A}(\boldsymbol{q_1},\boldsymbol{q_2})$, and making use of the standard Gaussian integral, we can obtain the desired result:

$$W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,z=0) = A'\sqrt{S(\boldsymbol{\rho}_1)S(\boldsymbol{\rho}_2)}\mu(\Delta\boldsymbol{\rho}) = A'\sqrt{\exp\left[-\frac{(\boldsymbol{\rho}_1)^2}{2\sigma_s^2}\right]\exp\left[-\frac{(\boldsymbol{\rho}_2)^2}{2\sigma_s^2}\right]}\exp\left[-\frac{(\boldsymbol{\Delta}\rho)^2}{2\sigma_\mu^2}\right] \tag{3}$$

with $\alpha = \sigma_s^2\left(\sigma_\mu^2 + 2\sigma_s^2\right)/(\sigma_\mu^2 + 4\sigma_s^2)$ and $\beta = 2\sigma_s^4/(\sigma_\mu^2 + 4\sigma_s^2)$, $\boldsymbol{\Delta}\rho = \boldsymbol{\rho}_2 - \boldsymbol{\rho}_1$.

(b) The cross-spectral density function at any z is given by

$$W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,z) = \frac{1}{k^4}\iint_{-\infty}^{\infty}\mathcal{A}(\boldsymbol{q_1},\boldsymbol{q_2})e^{-i\boldsymbol{q_1}\cdot\boldsymbol{\rho}_1+i\boldsymbol{q_2}\cdot\boldsymbol{\rho}_2}e^{-i\frac{q_1^2-q_2^2}{2k}z}d^2\boldsymbol{q_1}d^2\boldsymbol{q_2}.$$

Substituting the expression for the angular coherence function $\mathcal{A}(\boldsymbol{q_1},\boldsymbol{q_2})$, and again making use of the standard Gaussian integral, we can evaluate the integral to yield

$$W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,z) = A''\sqrt{S(\boldsymbol{\rho}_1)S(\boldsymbol{\rho}_2)}\mu(\Delta\boldsymbol{\rho}) = A''\sqrt{\exp\left[-\frac{(\boldsymbol{\rho}_1)^2}{2\sigma_s(z)^2}\right]\exp\left[-\frac{(\boldsymbol{\rho}_2)^2}{2\sigma_s(z)^2}\right]}\exp\left[-\frac{(\boldsymbol{\Delta}\rho)^2}{2\sigma_\mu(z)^2}\right] \tag{4}$$

where $\Delta\boldsymbol{\rho}_p = \boldsymbol{\rho}_{p1} - \boldsymbol{\rho}_{p2}$, and where

$$\sigma_s(z) = \frac{z\sqrt{\sigma_\mu^2 + 4\sigma_s^2}}{2k_0\sigma_s\sigma_\mu} \tag{5}$$

is the rms beam radius of the field at z in the far field; and

$$\sigma_\mu(z) = \frac{z\sqrt{\sigma_\mu^2 + 4\sigma_s^2}}{2k_0\sigma_s^2} \tag{6}$$

is the rms spatial coherence width of the field at z in the far field.

Solution 3.3

(a) The cross-spectral density function at any z is given by

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \frac{1}{k^4}\iint_{-\infty}^{\infty} \mathcal{A}(\boldsymbol{q_1}, \boldsymbol{q_2}) e^{-i\boldsymbol{q_1}.\boldsymbol{\rho}_1 + i\boldsymbol{q_2}.\boldsymbol{\rho}_2} e^{-i\frac{q_1^2 - q_2^2}{2k}z} d^2\boldsymbol{q_1} d^2\boldsymbol{q_2}.$$

Let us take the angular correlation function to be $\mathcal{A}(\boldsymbol{q_1}, \boldsymbol{q_2}) = \mathcal{S}(\boldsymbol{q_1})\delta(\boldsymbol{q_1} - \boldsymbol{q_2})$. Substituting it in the above integral we get

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \frac{1}{k^4}\int_{-\infty}^{\infty} \mathcal{S}(\boldsymbol{q_1}) e^{i\boldsymbol{q_1}.(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1)} d^2\boldsymbol{q_1} = \frac{1}{k^4}\int_{-\infty}^{\infty} \mathcal{S}(\boldsymbol{q_1}) e^{i\boldsymbol{q_1}.\boldsymbol{\Delta}\rho} d^2\boldsymbol{q_1}.$$

We find that the cross-spectral density does not depend on z. Therefore, we conclude that the transverse coherence width also does not depend on z.

Solution 3.4

(a) The amplitude of the field after the slits in a given mode index l is given by

$$\begin{aligned}
\alpha_l = &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(\phi)V(\phi)e^{-il\phi}d\phi \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\phi - \phi_1)V(\phi)e^{-il\phi}d\phi + \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\phi - \phi_2)V(\phi)e^{-il\phi}d\phi \\
&= \frac{1}{2\pi}V(\phi_1)e^{-il\phi_1} + \frac{1}{2\pi}V(\phi_2)e^{-il\phi_2} \\
&= \frac{1}{2\pi}\sum_{l'} \alpha_{l'} e^{il'\phi_1} e^{-il\phi_1} + \frac{1}{2\pi}\sum_{l'} \alpha_{l'} e^{il'\phi_2} e^{-il\phi_2} \\
&= \frac{1}{2\pi}\sum_{l'} \alpha_{l'} e^{i(l'-l)\phi_1} + \frac{1}{2\pi}\sum_{l'} \alpha_{l'} e^{i(l'-l)\phi_2}
\end{aligned}$$

Now, using, $\langle \alpha_l^* \alpha_{l'} \rangle = C_l \delta_{ll'}$, we get

$$\begin{aligned}
|\alpha_l|^2 = \langle \alpha_l^* \alpha_l \rangle &= \frac{1}{(2\pi)^2}\sum_{l'} C_{l'} + \frac{1}{(2\pi)^2}\sum_{l'} C_{l'} + \frac{1}{(2\pi)^2}\sum_{l'} C_{l'} e^{-i(l'-l)(\Delta\phi)} + \text{c.c.} \\
&= \frac{2}{(2\pi)^2}\sum_{l'} C_{l'}\left[1 + \cos[(l' - l)\Delta\phi]\right]
\end{aligned}$$

(b) Now, when $C_l = 1$, for $l = l_0$, and $C_l = 0$, otherwise, we get

$$|\alpha_l|^2 = \frac{1}{2\pi^2}[1 + \cos[(l_0 - l)\Delta\phi]$$

Solution 3.5

(a) The concept of pure and mixed states is very frequently encountered in Quantum Mechanics. The classical analog of the pure states in optics is a fully coherent state. The classical analog of the mixed state in optics is a partially coherent field, such as the field of a continuous-wave laser. A system is said to be in the pure state if it can be described by a single wave-function. This single wave-function can be in any arbitrary linear superposition of the eigenfunctions of the system. However, in the linear superposition, all the eigenfunctions have to have a definite phase relationship between them. On the other hand, a mixed state is a statistical mixture of several pure states. Since it is a statistical mixture it cannot have a wave-function representation and it can only be described through the density-matrix formalism.

(b) The difference between a coherent state and an incoherent state is same as the difference between a pure state and a mixed state. A coherent state is essentially a pure state and an incoherent state is essentially a mixed state. More specifically, in a coherent state, the fields at different space-time points have a fixed phase relationship between them. However, in an incoherent state the fields at different space-time points are not completely correlated in their phases. Yes, a coherent state is indeed a pure state.

(c) A coherent-mode representation of a partially-coherent field is the representation of the field in terms of field-modes that are completely coherent. One needs it because it is relatively much easier to describe the dynamics of several pure states and then add them up than to describe the dynamics of a single incoherent state, as it is. Diagonalizing a density matrix is the same as finding the coherent mode representation. There is a definite relationship between the number of modes in the coherent-mode representation and the degree of coherence, which is that the degree of coherence is inversely proportional to the number of modes in a coherent-mode representation.

(d) There is only one mode in the coherent-mode representation of a fully coherent field. An example is a monochromatic plane-wave field. Another example is a pulsed laser field in which all the frequency components have been phase-locked.

(e) For a completely incoherent field, the number of modes is equal to the dimensionality of of the basis. For example, in the momentum basis, a completely incoherent field will have infinite number of modes while in a polarization basis the number of modes will be two.

Solution 3.6

A completely unpolarized polarization density matrix $\mathrm{J}^{\mathrm{unpol}}$ and a completely polarized polarization matrix $\mathrm{J}^{\mathrm{pol}}$, are written as

$$\mathrm{J}^{\mathrm{unpol}} = A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \qquad \mathrm{J}^{\mathrm{pol}} = \begin{pmatrix} B & D \\ D^* & C \end{pmatrix}, \tag{7}$$

where $A \geq 0$, $B \geq 0$ and $C \geq 0$. Also $BC - DD^* = 0$.

We need to prove that

$$J = \begin{pmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{pmatrix} = \mathrm{J}^{\mathrm{pol}} + \mathrm{J}^{\mathrm{unpol}}$$

is a unique decomposition. Equating the left and the right hand side of the above equation, we obtain the conditions that

$$\begin{aligned} A + B &= J_{xx}, \\ A + C &= J_{yy}, \\ D &= J_{xy}, \\ D^* &= J_{yx}. \end{aligned}$$

Substituting for B and C from the above conditions in $BC - DD^* = 0$, we get

$$(J_{xx} - A)(J_{yy} - A) - J_{xy}J_{yx} = 0.$$

This is also the eigenvalue equation of matrix J, solving which, we obtain two solutions for A

$$A_1 = \frac{\text{tr J} + \sqrt{(\text{tr J})^2 - 4\text{det J}}}{2}$$
$$A_2 = \frac{\text{tr J} - \sqrt{(\text{tr J})^2 - 4\text{det J}}}{2}$$

The two solutions of A are the eigenvalues of matrix J. Let us take the second solution $A = A_2$. We find that in this case,

$$B = J_{xx} - A_2 = \frac{2J_{xx} - \text{tr J} + \sqrt{(\text{tr J})^2 - 4\text{det J}}}{2} = \frac{(J_{xx} - J_{yy}) + \sqrt{(J_{xx} - J_{yy})^2 + 4|J_{xy}|^2}}{2} \geq 0.$$

In a similar manner it can be shown that for $A = A_2$, $C \geq 0$. Thus $A = A_2$ is a valid solution. Next, we take $A = A_1$. In this case,

$$B = J_{xx} - A_1 = \frac{2J_{xx} - \text{tr J} - \sqrt{(\text{tr J})^2 - 4\text{det J}}}{2} = \frac{(J_{xx} - J_{yy}) - \sqrt{(J_{xx} - J_{yy})^2 + 4|J_{xy}|^2}}{2} \leq 0.$$

So, B is negative. Again, C can also be shown to be negative. However, for a valid solution the terms B and C need to be positive. Therefore, $A = A_1$ is not a valid solution. Thus, we have proved that the above decomposition is unique.

Solution 3.7

(a) A general polarization matrix is written as

$$J = \begin{pmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{pmatrix}$$

The determinant of the matrix is given by $\text{det}J = J_{xx}J_{yy} - J_{xy}J_{yx}$. We have

$$J_{xy}J_{yx} = \sqrt{J_{xx}J_{yy}}j_{xy} \times \sqrt{J_{xx}J_{yy}}j_{yx} = J_{xx}J_{yy}j_{xy}j_{xy}^* = J_{xx}J_{yy}|j_{xy}|^2$$

The quantity j_{xy} is the polarization degree of coherence and $0 \leq |j_{xy}| \leq 1$. Thus we have

$$\begin{aligned} & J_{xx}J_{yy} \geq J_{xy}J_{yx} \\ \text{or,} \quad & J_{xx}J_{yy} - J_{xy}J_{yx} \geq 0 \\ \text{or,} \quad & \text{det}J \geq 0 \end{aligned}$$

(b) The polarization matrix can be written in terms of the Stokes parameters as

$$J = \begin{pmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} S_0 + S_1 & S_2 + iS_3 \\ S_2 - iS_3 & S_0 - S_1 \end{pmatrix}$$

The two sides of the above equation can be equated to give

$$J_{xx} = \frac{1}{2}(S_0 + S_1)$$
$$J_{yy} = \frac{1}{2}(S_0 - S_1)$$
$$J_{xy} = \frac{1}{2}(S_2 + iS_3)$$
$$J_{yx} = \frac{1}{2}(S_2 - iS_3)$$

Now, we have

$$\begin{aligned}
& \det J \geq 0 \\
\text{or,}\quad & J_{xx}J_{yy} - J_{xy}J_{yx} \geq 0 \\
\text{or,}\quad & \frac{1}{4}[(S_0+S_1))(S_0-S_1) - (S_2+iS_3)(S_2-iS_3)] \geq 0 \\
\text{or,}\quad & S_0^2 - S_1^2 - (S_2^2+S_3^2) \geq 0 \\
\text{or,}\quad & S_0^2 \geq S_1^2 + S_2^2 + S_3^2
\end{aligned}$$

(c)

$$P = \left[1 - \frac{4\det J}{(\mathrm{tr}J)^2}\right]^{1/2} = \left[1 - \frac{S_0^2 - S_1^2 - S_2^2 - S_3^2}{S_0^2}\right]^{1/2} = \left[\frac{S_1^2+S_2^2+S_3^2}{S_0^2}\right]^{1/2}$$

Solution 3.8

(a) The electric field corresponding to the first laser, which is polarized along $\hat{x}$ direction can be written as $\boldsymbol{E}_1 = E_{1x}\hat{x} + E_{1y}\hat{y} = \hat{x}$. Thus $E_{1x} = 1$ and $E_{1y} = 0$. Similarly, the electric field corresponding to the first laser, which is polarized along $\hat{\theta}$ direction can be written as $\boldsymbol{E}_1 = E_{1x}\hat{x} + E_{1y}\hat{y} = \cos\theta\hat{x} + \sin\theta\hat{y}$ and thus $E_{1x} = \cos\theta$ and $E_{1y} = \sin\theta$. The polarization matrix corresponding to these two fields are given by

$$J_1 = \begin{pmatrix} E_1x^*E_1x & E_1x^*E_1y \\ E_1y^*E_1x & E_1y^*E_1y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad J_2 = \begin{pmatrix} E_2x^*E_2x & E_2x^*E_2y \\ E_2y^*E_2x & E_2y^*E_2y \end{pmatrix} = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$$

Now, since the two lasers are independent of each other the two fields will mix in an incoherent manner. Therefore, the resultant field is a partially polarized field and is given by a polarization matrix J, which is the sum of the two polarization matrices, that is,

$$J = J_1 + J_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix} = \begin{pmatrix} 1+\cos^2\theta & \cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$$

The degree of polarization is therefore given by

$$P = \left[1 - \frac{4\det J}{(\mathrm{tr}J)^2}\right]^{1/2} = \left[1 - \frac{4[(1+\cos^2\theta)\sin^2\theta - \cos^2\theta\sin^2\theta]}{(1+\cos^2\theta+\sin^2\theta)^2}\right]^{1/2} = \left[\frac{4-4\sin^2\theta}{4}\right]^{1/2} = |\cos\theta|$$

(b) When $\theta = 0$, $P = 1$, implying perfectly polarized light.

(c) When $\theta = \pi/2$, $P = 0$, implying perfectly unpolarized light