Solutions: Homework # 4

Solution 4.1: One-Photon Interference (2+2+4+2+2+2+2+2+2=20 marks)

(a) The Electric field at the detector is

$$\hat{E}^{(+)}(t) = k_1\hat{E}^{(+)}(t-t_1) + k_2\hat{E}^{(+)}(t-t_2)$$

Therefore, the probability is

$$\begin{aligned}P_A &= \langle\langle\psi|\hat{E}^{(-)}(t)\hat{E}^{(+)}(t)|\psi\rangle\rangle_e \\ &= \langle\langle\psi|[k_1^*\hat{E}^{(-)}(t-t_1) + k_2^*\hat{E}^{(-)}(t-t_2)][k_1\hat{E}^{(+)}(t-t_1) + k_2\hat{E}^{(+)}(t-t_2)]|\psi\rangle\rangle_e\end{aligned}$$

Expanding and simplifying the above expression, we get

$$\begin{aligned}P_A = |k_1|^2\langle\langle\psi|\hat{E}^{(-)}(t-t_1)\hat{E}^{(+)}(t-t_1)|\psi\rangle\rangle_e + |k_2|^2\langle\langle\psi|\hat{E}^{(-)}(t-t_2)\hat{E}^{(+)}(t-t_2)|\psi\rangle\rangle_e \\ + 2\mathrm{Re}k_1^*k_2\langle\langle\psi|\hat{E}^{(-)}(t-t_1)\hat{E}^{(+)}(t-t_2)|\psi\rangle\rangle_e\end{aligned}$$

(b) Since $\hat{E}^{(-)}(t)$ is the operator representing the negative frequency part of the field and is therefore defined as

$$\hat{E}^{(-)}(t) = \int_{-\infty}^{0} \hat{a}(\omega)e^{-i\omega t}d\omega$$

Substituting $\omega' \to \omega$, we get

$$\hat{E}^{(-)}(t) = \int_{\infty}^{0} \hat{a}(-\omega')e^{i\omega' t}(-d\omega') = \int_{0}^{\infty} \hat{a}(-\omega')e^{i\omega' t}d\omega'$$

Since $\hat{E}(t)$ is a Hermitian operator, we have $\hat{a}(-\omega') = \hat{a}^\dagger(\omega')$ and therefore

$$\hat{E}^{(-)}(t) = \int_{0}^{\infty} \hat{a}^\dagger(\omega)e^{i\omega t}d\omega.$$

(c)

$$\hat{E}^{(+)}(t-t_2)|\psi\rangle = \int_0^\infty \hat{a}(\omega)e^{-i\omega(t-t_2)}d\omega\int_0^\infty V(\omega')|\omega'\rangle d\omega' = \iint_0^\infty V(\omega')\hat{a}(\omega)|\omega'\rangle e^{-i\omega(t-t_2)}d\omega d\omega'$$

We know that $\hat{a}(\omega)|\omega'\rangle = |\mathrm{vac}\rangle\delta(\omega-\omega')$. Using this, we get

$$\hat{E}^{(+)}(t-t_2)|\psi\rangle = \iint_0^\infty V(\omega')|\mathrm{vac}\rangle\delta(\omega-\omega')e^{-i\omega(t-t_2)}d\omega d\omega' = \int_0^\infty V(\omega)e^{-i\omega(t-t_2)}|\mathrm{vac}\rangle d\omega$$

Therefore, we can now write

$$\langle\langle\psi|\hat{E}^{(-)}(t-t_1)\hat{E}^{(+)}(t-t_2)|\psi\rangle\rangle_e = \iint_0^\infty \langle V^*(\omega')V(\omega)\rangle_e e^{i\omega'(t-t_1)}e^{-i\omega(t-t_2)}d\omega d\omega'$$

For a stationary random source we have $\langle V^*(\omega')V(\omega)\rangle_e = S(\omega)\delta(\omega-\omega')$ and

$$\begin{aligned}\langle\langle\psi|\hat{E}^{(-)}(t-t_1)\hat{E}^{(+)}(t-t_2)|\psi\rangle\rangle_e &= \iint_0^\infty S(\omega)\delta(\omega-\omega')e^{i\omega'(t-t_1)}e^{-i\omega(t-t_2)}d\omega d\omega' \\ &= \int_0^\infty S(\omega)e^{-i\omega\tau}d\omega,\end{aligned}$$

where $\tau = t_1 - t_2$.

(d) The field is quasi-monochromatic, that is, the frequency-bandwidth $\Delta\omega$ of the field is much small compared to the central frequency ω_0 of the field: $\Delta\omega \ll \omega_0$. So, let's substitute $\omega = \omega' + \omega_0$. We then get

$$\langle\langle\psi|\hat{E}^{(-)}(t-t_1)\hat{E}^{(+)}(t-t_2)|\psi\rangle\rangle_e = \int_{-\omega_0}^{\infty} S(\omega'+\omega_0)e^{-i(\omega'+\omega_0)\tau}d\omega', \tag{1}$$

$$= e^{-i\omega_0\tau}\int_{-\omega_0}^{\infty} S(\omega'+\omega_0)e^{-i\omega'\tau}d\omega' \tag{2}$$

$$= e^{-i\omega_0\tau}\int_{-\infty}^{\infty} S_0(\omega)^{-i\omega\tau}d\omega, \tag{3}$$

where $S_0(\omega) = S(\omega+\omega_0)$.

(e) Using the expression derived above, and substituting $\tau = 0$, we get

$$\langle\langle\psi|\hat{E}^{(-)}(t-t_1)\hat{E}^{(+)}(t-t_1)|\psi\rangle\rangle_e = \langle\langle\psi|\hat{E}^{(-)}(t-t_2)\hat{E}^{(+)}(t-t_2)|\psi\rangle\rangle_e = \int_{-\infty}^{\infty} S_0(\omega)d\omega = I \tag{4}$$

Substituting from Eqs. (**??**) and (**??**) into Eq. (**??**), we obtain the desired form for the one-photon interference law.

$$P_A = |k_1|^2 I + |k_1|^2 I + 2|k_1||k_2|I|\gamma(\tau)|\cos(\omega_0\tau + \phi).$$

(f) In the classical treatment, the interference is always explained in terms of the division of field-amplitude. The use of analytic signal to separate out the positive and negative frequency parts of the field is only for mathematical convenience; one actually only deals with real field in the classical treatment. On the other hand, in the quantum treatment the negative and positive frequency parts of the field have definite meaning in terms of creation and annihilation operators. when it comes to explaining interference of photons at a single photon level, the classical description becomes inadequate since a single photon cannot be divided. So, while the classical description based on division of amplitude implies division of energy as well, the quantum description based on division of wavefunction does not imply division of energy.

(g) We have

$$\gamma(\tau) = \frac{\int_{-\infty}^{\infty} S_0(\omega)e^{-i\omega\tau}d\omega}{\int_{-\infty}^{\infty} S_0(\omega)d\omega}.$$

Using the Gaussian integrals derived during the previous homework, we can show that

$$\int_{-\infty}^{\infty} S_0(\omega)e^{-i\omega\tau}d\omega = e^{-\tau^2\Delta\omega^2/2}$$

$$\int_{-\infty}^{\infty} S_0(\omega)d\omega = 1.$$

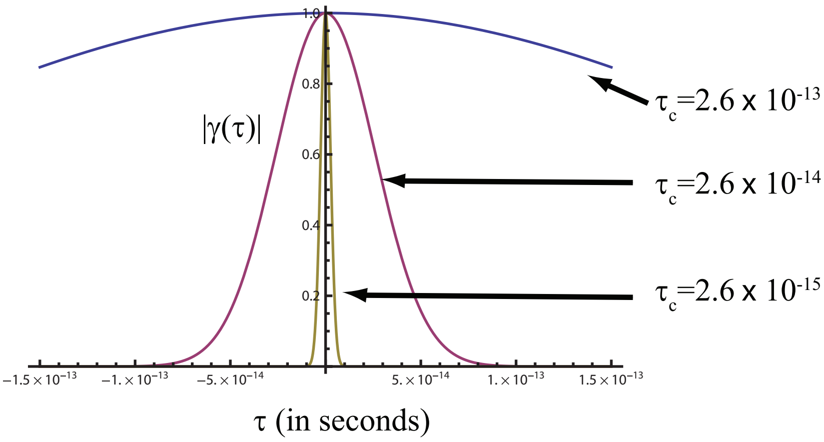
Therefore, $\gamma(\tau) = e^{-\tau^2\Delta\omega^2/2}$. The Standard deviation of this distribution is $1/\Delta\omega$ and thus the coherence time is $1/\Delta\omega$.

(h) When $\tau \gg \tau_c$, $\gamma(\tau) \to 0$. This mean that the interference is no longer there. Classically, it means that the two fields reaching the detector through the two alternatives have become incoherent. This is because once the field points are separated out by more than the coherence time there no longer remains any correlation between the electric field vibrations at the two points. Quantum mechanically, this means that photons arriving at the detector through the two alternatives have now become distinguishable, that is, it is indeed possible in principle to find out which alternative a photon came through.

(i)

$$\omega = \frac{2\pi c}{\lambda}; \qquad d\omega = -\frac{2\pi c \Delta\lambda}{\lambda^2}$$

$\Delta\lambda$ (in nm)	$\Delta\omega$ (in Hz)	τ_c (in seconds)
1	3.84×10^{12}	2.6×10^{-13}
10	3.84×10^{13}	2.6×10^{-14}
100	3.84×10^{14}	2.6×10^{-15}



(j) In a continuous-wave field, the electric fields at different frequencies remain completely uncorrelated. Because of the presence of multiple uncorrelated frequencies, the correlation in temporal domain exists only over a certain range, which is called the coherence time.

If the electric field at different frequencies remain correlated then the entire field remains completely correlated, irrespective of the frequency-bandwidth of the field. One such example is the pulsed laser-field in which all the different frequencies are phase-locked.

Solution 4.2: Hong-Ou-Mandel Effect (5+5+10=20 marks)

The expression for coincidence count rate R_{si} of two detectors R_s and R_i is given by:

$$R_{si} = R_1 + R_2 + 2\sqrt{R_1 R_2}\gamma(\Delta L)\gamma'(\Delta L')\cos(k_0\Delta L + k_d\Delta L' + \Delta\phi) \tag{5}$$

The expression for the one-photon detection probability R_X at detector D_X in a two-photon interference experiment to be:

$$R_X = \sum_{i=1}^{N} R_{XY_i} \tag{6}$$

where R_{XY_i} is the coincidence count rate of detectors D_X and D_{Y_i}, and Y_i is the set of all possible locations where the entangled partner of the photon detected at X can go to.

(a) Refer to figure **??**(a) and (b). We find that $\Delta L = 0, \Delta L' = 4x$ and $\Delta\phi = \pi$. Also, using $R_1 = R_2 = R$ and substituting in the formula for coincidence, we get:

$$R_{si} = 2R[1 - \gamma'(4x)]$$

(b) In order to calculate the count rate at individual detectors, we need to use the formula for one-photon detection probability. Now for a photon arriving at detector D_s, the entangled pair can either go to the detector D_i or

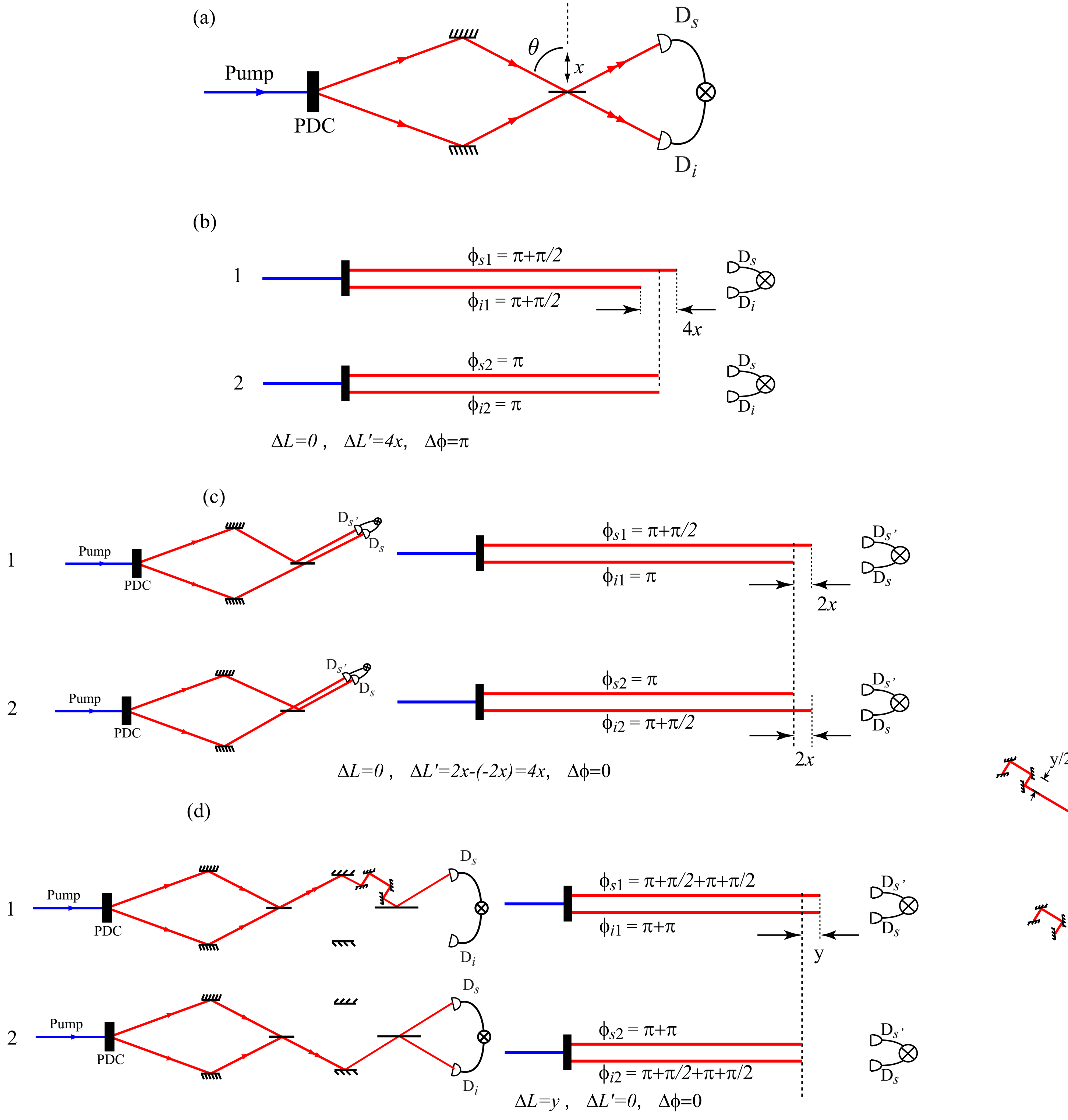


FIG. 1: Hong-Ou-Mandel Setup

to the detector D'_s. Please note that to calculate the one-photon count rate, we'll have to consider detector D'_s because in principle the entangled pair can go there. Therefore the one-photon count rate at detector D_s will be given by:

$$R_s = R_{si} + R_{ss'}$$

Fig. ??(c) has the two alternatives that will lead to a coincidence of detectors D_s and $D_{s'}$. From the two-photon path diagram, we get $\Delta L = 0, \Delta L' = 4x$ and $\Delta\phi = 0$. Again, using $R_1 = R_2 = R$, and substituting in Eq. (??), we get

$$R_{si} = 2R[1 + \gamma'(4x)]$$

Therefore

$$R_s = R_{si} + R_{ss'} = 4R,$$

that is the one-photon detection probability shows no interference at D_i. We get the same result for the one-photon detection probability at D_i, that is, $R_i = 4R$.

(c) Fig. ??(d) shows the two alternatives that will lead to the coincidence counts of the two detectors. We find that at $x = 0$, $\Delta L = y, \Delta L' = 0$ and $\Delta\phi = 0$. Therefore, we get using Eq. (??) the two photon count rate R_{si} to be

$$R_{si} = 2R[1 + \gamma(y)\cos(k_0 y)] = 2R[1 + \gamma(y)\cos(2\frac{2\pi}{\lambda_{s0}}y)],$$

where $\lambda_{s0=}$ is the wavelength of each of the signal and idler photons. Thus we see that the fringe period is $2\pi/(2k_{s0}) = \lambda_{s0}/2$. Note that if we had used a classical light at λ_{s0}, instead of the two-photon $N00N$ state at λ_{s0}, the fringe period would be λ_{s0} only. In fact, if we use the N-photon $N00N$ state, the fringe period would be λ_{s0}/N.

Solution 4.3: Double-pass experiment (10+5+5+10+5=35 marks)

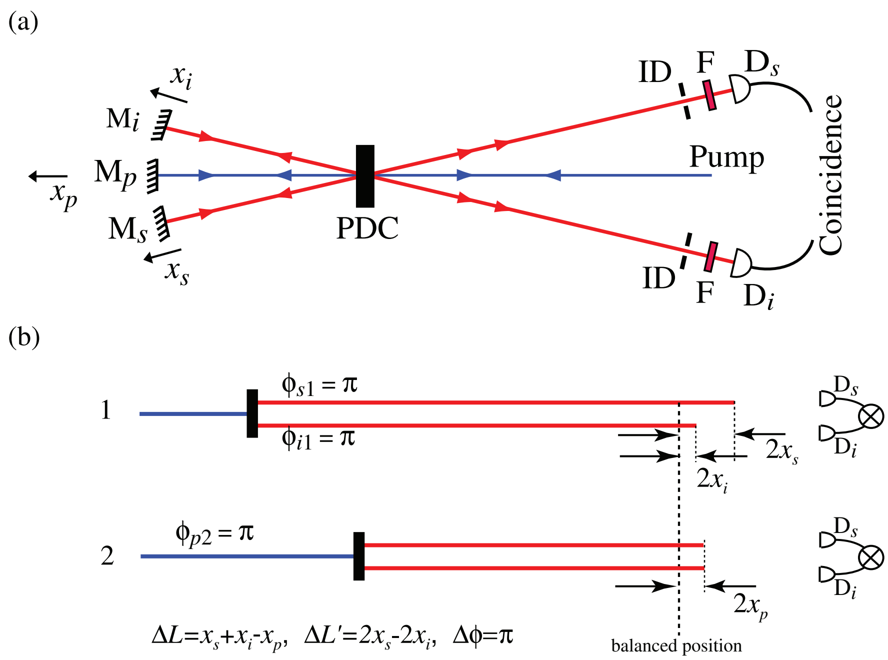


FIG. 2: Double-pass Setup

(a) Refer to Figure ??(b). We find that $\Delta L = x_s + x_i - 2x_p$, $\Delta L' = 2x_s - 2x_i$, and $\Delta\phi = \pi$. Therefore, using Eq. (??), we find the coincidence count rate R_{si} to be

$$R_{si} = 2R[1 - \gamma(x_s + x_i - 2x_p)\gamma'(2x_s - 2x_i)\cos[k_0(x_s + x_i - 2x_p)]]$$

Also, for every photon arriving at D_s, the only place that the entangled partner can go to is the idler detector. Similarly, for every photon arriving at D_i, the entangled partner can only go to the signal detector. Thus, we get

$$R_s = R_i = R_{si}$$

So, we find that whatever interference profile we see in coincidence will be seen in the photon count rates of the individual detectors.

(b) We need to make $\Delta L' = 2x_s - 2x_i = 0$, that is, $x_s = x_i$. This means that we need to move both the signal and idler mirrors in the same direction and with the same amplitude.

(c) For this, we need $\Delta L = x_s + x_i = 0$, that is, $x_s = -x_i$. This means that we need to move both the signal idler mirrors in the opposite directions but with the same amplitude.

(d) In order to see a dip in the coincidence $\Delta L = x_s + x_i - 2x_p = 0$, such that the coincidence count rate is given by

$$R_{si} = 2R[1 - \gamma'(2x_s - 2x_i)]$$

Here the plot of R_{si} versus $2x_s - 2x_i$ is in the form of a dip profile, which is the Hong-Ou-Mandel type profile. In order to get a hump one needs to increase x_p by λ such that $\Delta L = x_s + x_i - 2x_p = \lambda$. And therefore,

$$R_{si} = 2R[1 - \gamma'(2x_s - 2x_i)\cos[k_0(\pi)]] = 2R[1 + \gamma'(2x_s - 2x_i)]$$

The above profile is a hump profile. These dip/hump profile cannot be explained in terms of the photon-bunching since there is no mixing of photons at a beam splitter in the double-pass setup.

(e) In this experiment, the entangled photon have only one choice, which is that one of them go the signal detector and the other one go to the idler detector. There are no other possibilities. However the photon-number needs to be conserved and this requires that a modulation in the coincidence count is followed by an equal and opposite modulations in the pump photon count. As a result, when the coincidence count rate is at its minimum, no pump photon gets down-converted.

Solution 4.4: Two-photon Coherence function (10+5+10=25 marks)

(a) Refer to Fig. **??**(b). Since the coincidence detection system is much faster than the time delay between the two arms of the interferometer, there will be only two alternatives that will contribute to the coincidence detection. For these two alternatives $\Delta L = x$, $\Delta L' = 0$, and $\Delta\phi = 5\pi$. Therefore the coincidence count rate is

$$R_{si} = 2R[1 - \gamma(x)\cos(k_0 x)]$$

Here we have made one other assumption that $x << l^p_{\text{coh}}$. For calculating the one-photon count rate R_s, we note that $R_s = R_{si} + R_{ss'}$. Refer to Fig. **??**(c). We find the for $R_{ss'}$, $\Delta L = x$, $\Delta L' = 0$, and $\Delta\phi = 6\pi$. Therefore,

$$\begin{aligned} R_{ss'} &= 2R[1 + \gamma(x)\cos(k_0 x)], \\ \text{And thus} \quad R_s &= R_{ss} + R_{ss'} = 4R \end{aligned}$$

We see that the one-photon count rates of the two detectors do not show any interference pattern.

(b) When a minimum is observed in the coincidence count rate, the entangled photons both leave through either one of the output ports of the interferometer.

(c) Figure **??** represent one another possible setup for making a two-photon $N00N$ state. We can see that the state of the two-photons in this setup is : $|\psi\rangle = \frac{1}{\sqrt{2}}[|2\rangle_{k_1}|0\rangle_{k_2} + |0\rangle_{k_1}|2\rangle_{k_2}]$

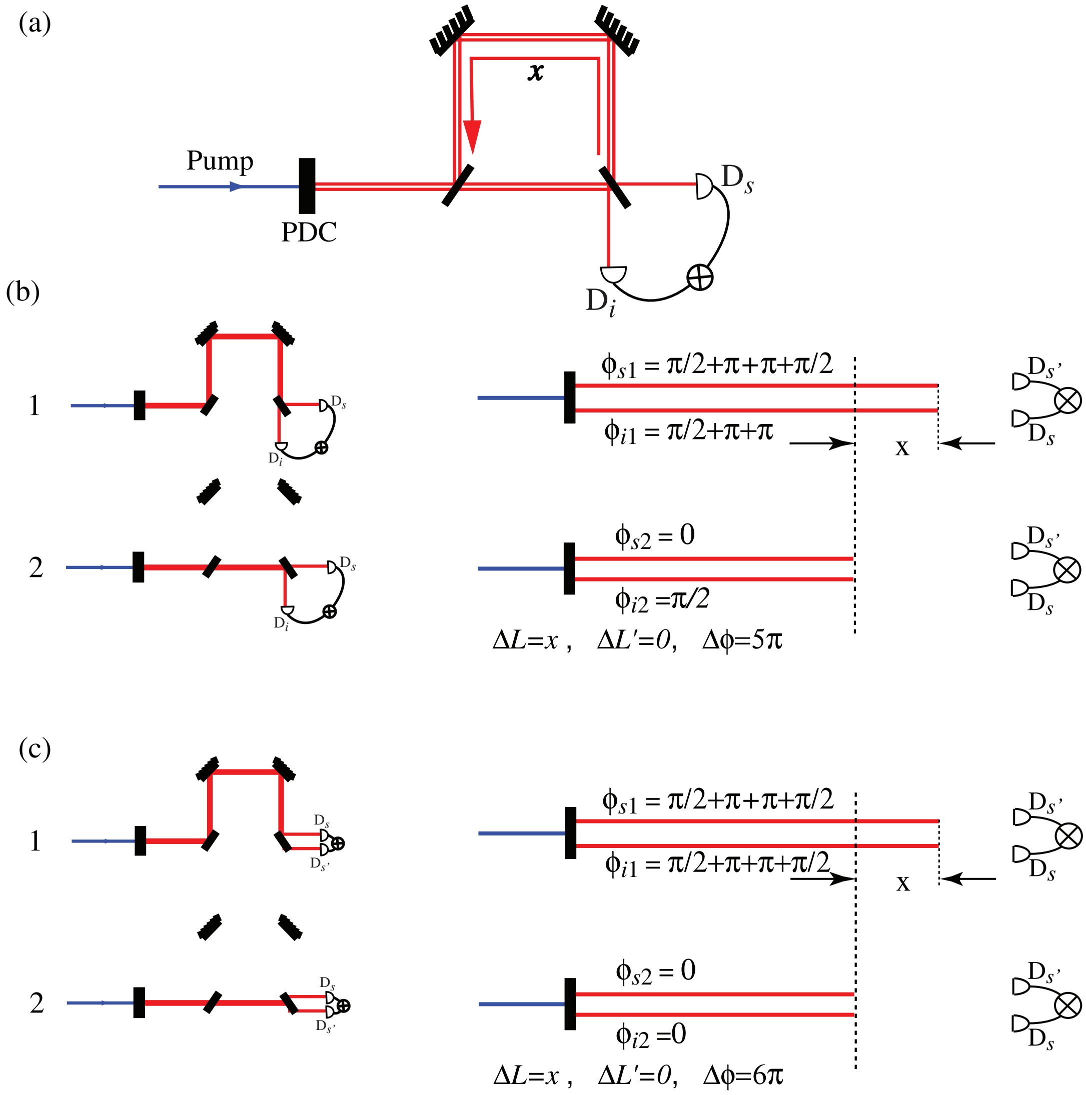


FIG. 3: Two-photon Michaelson Interferometer

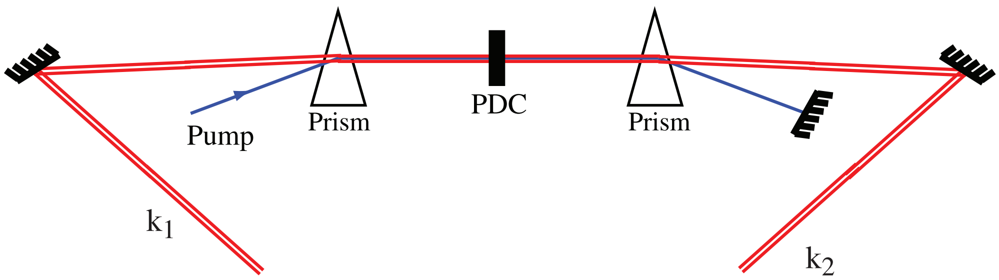


FIG. 4: Two-photon Michaelson Interferometer