Solutions: Homework # 5

Solutions 5.1: Misc Questions (5+5=10 marks)

(a) Two-photon coherence refers to the ability with which one could predict the field vibrations at a pair of space-time points within the field, given the field vibrations at some reference pair of space-time points. Again, experimentally, two-photon coherence refers to the ability of a field to give rise to two-photon interference.

(b) This means that if the orbital angular momentum of the photon is measured, one would find the photon to have the orbital angular momentum $l\hbar$ with probability $|c_l|^2$

Solutions 5.2: Spatial two-photon wave-function (5+20+5+10=40 marks)

(a) The signal electric field operator at the detectors can be written as

$$\hat{E}_s^{(+)}(\boldsymbol{r}_s) = \int d\boldsymbol{q}\hat{a}_s(\boldsymbol{q})e^{i(\boldsymbol{q}\cdot\boldsymbol{\rho}_s - k_z z)}$$

We consider only the collinear down-conversion and thus make paraxial approximations. Therefore, we can write

$$k_z = \sqrt{k_s^2 - q^2} = k_s\sqrt{1-\frac{q^2}{k_s^2}} \approx k_s\sqrt{1-\frac{q^2}{2k_s^2}} = k_s - \frac{q^2}{2k_s}$$

Substituting this is the above equation and doing a similar approximation for the idler field we get,

$$\hat{E}_s^{(+)}(\boldsymbol{r}_s) = e^{ik_s z}\int d\boldsymbol{q}\hat{a}_s(\boldsymbol{q})e^{i(\boldsymbol{q}\cdot\boldsymbol{\rho}_s - q^2 z/2k_s)},$$
$$\hat{E}_i^{(+)}(\boldsymbol{r}_i) = e^{ik_i z}\int d\boldsymbol{q'}\hat{a}_i(\boldsymbol{q'})e^{i(\boldsymbol{q'}\cdot\boldsymbol{\rho}_i - q'^2 z/2k_i)},$$

(b) For the one-dimensional version of this problem, we have

$$|\psi_{\rm tp}\rangle = A\iint_{\infty}^{\infty} dk_{sx}dk_{ix}V(k_{sx}+k_{ix})|k_{sx}\rangle|k_{ix}\rangle$$
$$\hat{E}_s^{(+)}(x_s) = e^{ik_s z}\int dk_{sx}\hat{a}_s(k_{sx})e^{i(k_{sx}x_s - k_{sx}^2 z/2k_s)}$$

We need to find the two-photon probability amplitude $\psi_{\rm tp}(x_s, x_i) = \langle{\rm vac}|\hat{E}_i^{(+)}(x_i)\hat{E}_s^{(+)}(x_s)|\psi_{\rm tp}\rangle$. Using $\hat{a}_s(k_{sx'})|k_{sx}\rangle = \delta(k_{sx}-k_{sx'})$ and $\hat{a}_i(k_{ix'})|k_{ix}\rangle = \delta(k_{ix}-k_{ix'})$, and integrating out the Dirac-delta functions we get,

$$\langle{\rm vac}|\hat{E}_i^{(+)}(x_i)\hat{E}_s^{(+)}(x_s)|\psi_{\rm tp}\rangle$$
$$= Ae^{i(k_s+k_i)z}\iint_{-\infty}^{\infty} dk_{sx}dk_{ix}V(k_{sx}+k_{ix})\exp\left[i(k_{sx}x_s + k_{ix}x_i) - i\left(\frac{k_{sx}^2}{2k_s}+\frac{k_{ix}^2}{2k_i}\right)\right]$$

Now assume degenerate down-conversion,that is, $k_s = k_i = k_0/2$. Also substitute for the Gaussian pump field: $V(k_{sx}+k_{ix}) = \exp\left[-\frac{w_0^2(k_{sx}+k_{ix})^2}{4}\right]$. We now get

$$\langle{\rm vac}|\hat{E}_i^{(+)}(x_i)\hat{E}_s^{(+)}(x_s)|\psi_{\rm tp}\rangle$$
$$= Ae^{ik_0 z}\int_{-\infty}^{\infty} dk_{ix}\exp\left[ik_{ix}x_i - i\frac{k_{ix}^2}{k_0} - \frac{w_0^2k_{ix}^2}{4}\right]\int_{-\infty}^{\infty} dk_{sx}\exp\left[ik_{sx}x_s - i\frac{k_{sx}^2}{k_0} - \frac{w_0^2k_{sx}^2}{4} - \frac{w_0^2k_{sx}k_{sx}}{z}\right]$$

Let's first solve the second integral. Using the Gaussian integration formula that we already derived in the earlier homework: $\int_{-\infty}^{\infty} e^{-\alpha x^2+\beta x}dx = (\pi/\alpha)^{1/2}e^{\beta^2/4\alpha}$, we get

$$\int_{-\infty}^{\infty} dk_{sx} \exp\left[ik_{sx}x_s - i\frac{k_{sx}^2}{k_0} - \frac{w_0^2 k_{sx}^2}{4} - \frac{w_0^2 k_{sx}k_{sx}}{z}\right]$$
$$= \left(\frac{\pi}{\alpha}\right)^{1/2} \exp\left[-\frac{x_s^2}{4\alpha}\right] \exp\left[\frac{w_0^4 k_{ix}^2 - 4iw_0^2 x_s k_{ix}}{16\alpha}\right]$$

where $\alpha = \left(\frac{iz}{k_0} + \frac{w_0^2}{4}\right)$. Putting the above integration result into the expression for two-photon probability amplitude we get,

$$\langle \mathrm{vac}|\hat{E}_i^{(+)}(x_i)\hat{E}_s^{(+)}(x_s)|\psi_{\mathrm{tp}}\rangle$$
$$= Ae^{ik_0 z}\left(\frac{\pi}{\alpha}\right)^{1/2} \exp\left[-\frac{x_s^2}{4\alpha}\right] \int_{-\infty}^{\infty} dk_{ix} \exp\left[ik_{ix}x_i - i\frac{k_{ix}^2}{k_0} - \frac{w_0^2 k_{ix}^2}{4} + \frac{w_0^4 k_{ix}^2 - 4iw_0^2 x_s k_{ix}}{16\alpha}\right]$$

Using the same Gaussian integral formula as above the above equation can be shown to be

$$\langle \mathrm{vac}|\hat{E}_i^{(+)}(x_i)\hat{E}_s^{(+)}(x_s)|\psi_{\mathrm{tp}}\rangle$$
$$= Ae^{ik_0 z}\left(\frac{\pi}{\alpha}\right)^{1/2} \exp\left[-\frac{x_s^2}{4\alpha}\right] \left(\frac{\pi}{\alpha - w_0^4/(16\alpha)}\right)^{1/2} \exp\left[-\frac{(x_i - w_0^2 x_s/(4\alpha))^2}{4(\alpha - w_0^4/(16\alpha))^2}\right]$$

The above expression can be simplified and be shown to be

$$\langle \mathrm{vac}|\hat{E}_i^{(+)}(x_i)\hat{E}_s^{(+)}(x_s)|\psi_{\mathrm{tp}}\rangle$$
$$= Ae^{ik_0 z}\left(\frac{16\pi^2}{16\alpha^2 - w_0^4}\right)^{1/2} \exp\left[-\frac{16\alpha^2(x_s^2+x_i^2) - 8\alpha w_0^2 x_i x_s}{4\alpha(16\alpha^2 - w_0^4)}\right]$$

This can be further simplified and rearranged to get he desired form

$$\langle \mathrm{vac}|\hat{E}_i^{(+)}(x_i)\hat{E}_s^{(+)}(x_s)|\psi_{\mathrm{tp}}\rangle$$
$$= Ae^{ik_0 z}\left(\frac{16\pi^2}{16\alpha^2 - w_0^4}\right)^{1/2} \exp\left[-\frac{(x_s+x_i)^2}{4w^2(z)}\right] \exp\left[\frac{i(x_s+x_i)^2 z}{2k_0 w_0^2 w^2(z)}\right] \exp\left[\frac{ik_0(x_s-x_i)^2}{8z}\right]$$

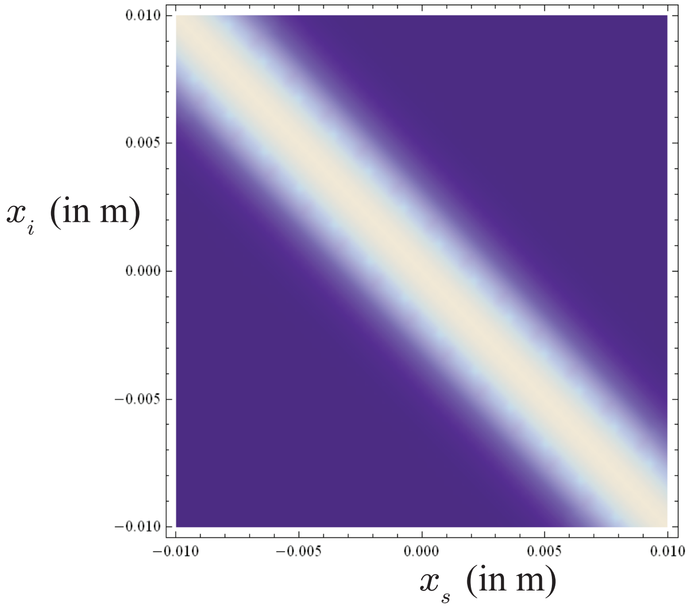
(c) The two photons probability is given by

$$\begin{aligned}
R(x_s, x_i) &= \langle\psi_{\mathrm{tp}}|\hat{E}_s^{(-)}(x_s)\hat{E}_i^{(-)}(x_i)\hat{E}_i^{(+)}(x_i)\hat{E}_s^{(+)}(x_s)|\psi_{\mathrm{tp}}\rangle \\
&= \langle\psi_{\mathrm{tp}}|\hat{E}_s^{(-)}(x_s)\hat{E}_i^{(-)}(x_i)|\mathrm{vac}\rangle\langle \mathrm{vac}|\hat{E}_i^{(+)}(x_i)\hat{E}_s^{(+)}(x_s)|\psi_{\mathrm{tp}}\rangle \\
&= \psi_{\mathrm{tp}}^{\dagger}(x_s, x_i)\psi_{\mathrm{tp}}(x_s, x_i) \\
&= |\psi_{\mathrm{tp}}(x_s, x_i)|^2 \to A\exp\left[-\frac{(x_s+x_i)^2}{2w^2(z)}\right]
\end{aligned}$$

(d) Figure shows the density plot of the probability distribution. The plot is for $\lambda = 800$ nm, $w_0 = 2$ mm, and $z = 1$ m. We find that at a particular z, given that the signal photon is detected at some x_s, the range over which the idler has a finite probability of getting detected is $w(z)$. This is the correlation width of the down-converted photons. We find that this correlation width is equal to the width of the Gaussian beam at z. As z increases, the width $w(z)$ increases and we find that the position-correlation width increases in the same proportion.

Solutions 5.3: Spatial Coherence and Entanglement (5+5+5+5+5+5=30 marks)

(a) The diagonal terms of the two-qubit density matrix are the probabilities with which the two photons are found in different two-photon alternatives. The two-photon spectral density $S^{(2)}(\boldsymbol{\rho}_1, z)$ is proportional to the probability



with which the two photons are found at the pair of the points $\boldsymbol{\rho}_{s1}$ and $\boldsymbol{\rho}_{i1}$. In the above problem, the qubits are being defined using the spatial locations of the photons and therefore the diagonal terms of the two-qubit density matrix are proportional to the two-photon photon spectral densities.

(b) The off-diagonal terms of the two-qubit density matrix refers to the coherence between the two two-photon alternatives that make the two-qubit state. The two-photon cross spectral density quantifies the correlations between the pair of space-time points $(\boldsymbol{\rho}_{s1}, \boldsymbol{\rho}_{i1})$ and $(\boldsymbol{\rho}_{s2}, \boldsymbol{\rho}_{i2})$. And therefore $c = d^* = \eta k_1 k_2 W^{(2)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z)$.

(c) For a given two-qubit density matrix ρ, the concurrence $C(\rho)$ is given by $C(\rho) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$. Here the λ_is are the eigenvalues, in descending order, of matrix $\zeta = \rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, with $\sigma_y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ being the usual Pauli operator and ρ^* the complex conjugate of ρ. For the density matrix ρ_{qubit}, the matrix ζ becomes

$$\zeta = \begin{pmatrix} ab + cd & 0 & 0 & 2ac \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2bd & 0 & 0 & ab + cd \end{pmatrix}.$$

The eigenvalues of ζ in descending order are:

$$\lambda_1 = (\sqrt{ab} + |c|)^2,$$
$$\lambda_2 = (\sqrt{ab} - |c|)^2,$$
$$\lambda_3 = 0 \text{ and}$$
$$\lambda_4 = 0;$$

where we have substituted $c = d^*$. Thus, the concurrence $C(\rho_{\mathrm{qubit}}) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$ is given by

$$C(\rho_{\mathrm{qubit}}) = 2|c| = \frac{2k_1 k_2 |W^{(2)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z)|}{k_1^2 S^{(2)}(\boldsymbol{\rho}_1, z) + k_2^2 S^{(2)}(\boldsymbol{\rho}_2, z)}.$$

We thus find that for a spatial two-qubit state, concurrence is proportional to the magnitude of the two-photon cross-spectral density at the two pairs of transverse positions that define the two-qubit state.

(d) The visibility V of the two-photon interference fringes is given by

$$V = \frac{R_{si}(\boldsymbol{r}_s, \boldsymbol{r}_i)_{\max} - R_{si}(\boldsymbol{r}_s, \boldsymbol{r}_i)_{\min}}{R_{si}(\boldsymbol{r}_s, \boldsymbol{r}_i)_{\max} + R_{si}(\boldsymbol{r}_s, \boldsymbol{r}_i)_{\min}}$$

The max and min correspond to the situations when the cosine term is +1 and -1, respectively. Thus we get:

$$V = \frac{2k_1 k_2 \sqrt{S^{(2)}(\boldsymbol{\rho}_1, z) S^{(2)}(\boldsymbol{\rho}_2, z)}}{k_1^2 S^{(2)}(\boldsymbol{\rho}_1, z) + k_2^2 S^{(2)}(\boldsymbol{\rho}_2, z)} \mu^{(2)}(\Delta\boldsymbol{\rho}, z).$$

Comparing the visibility with the concurrence we find that $V = C(\rho_{\text{qubit}})$.

(e) When $a = b$ and $|c| = |d| = 1/2$, we get $C(\rho_{\text{qubit}}) = 1$. This means that the two-qubit state is maximally entangled. This can be achieved when (i) the probabilities for both the two-photon alternatives that make the two-qubit state are the same and (ii) the two two-photon alternatives are completely coherent, that is, indistinguishable with one another.

(f) When $a = b$ and $|c| = |d| = 0$, we get $C(\rho_{\text{qubit}}) = 0$. This means that the two-qubit state is completely un-entangled. This is the situation whenever, the two two-photon alternatives that make the two-qubit state are completely incoherent, that is, distinguishable from one another.

Solutions 5.4: Concurrence of an X-matrix Spatial Coherence and Entanglement (2+6+2+10=20 marks)

(a) The dinsity matrix corresponding to $|\psi_1\rangle$ is

$$\rho_1 = |\psi_1\rangle\langle\psi_1| = \begin{pmatrix} a^2 & 0 & 0 & ab \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ ab & 0 & 0 & b^2 \end{pmatrix}$$

(b) The concurrence of two-qubit density matrix ρ is calculated as follow. Take the two-qubit density matrix ρ and construct the density matrix $\tilde{\rho}$ using the spin flip operation

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y),$$

where $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is the Pauli's matrix. Construct a new matrix $\rho\tilde{\rho}$ and find its eigenvalues and arrange them in descending order $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. The concurrence of the two-qubit state is then given as

$$C(\rho) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}.$$

For the above density matrix ρ_1, we have

$$\rho_1\tilde{\rho}_1 = \begin{pmatrix} 2a^2b^2 & 0 & 0 & 2a^3b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2ab^3 & 0 & 0 & 2a^2b^2 \end{pmatrix}$$

The eigenvalues of this matrix in the descending order are $(4a^2b^2, 0, 0, 0)$. Therefore the concurrence is $2|ab|$.

(c) The density matrix in this case is

$$\rho = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2| = \frac{1}{2}\begin{pmatrix} a^2 & 0 & 0 & ab \\ 0 & p^2 & pq & 0 \\ 0 & pq & q^2 & 0 \\ ab & 0 & 0 & b^2 \end{pmatrix}$$

(d) In order to calculate the concurrence we first calculate $\tilde{\rho}$, which can be shown to be

$$\rho\tilde{\rho} = \frac{1}{2}\begin{pmatrix} a^2b^2 & 0 & 0 & a^3b \\ 0 & p^2q^2 & p^3q & 0 \\ 0 & pq^3 & p^2q^2 & 0 \\ ab^3 & 0 & 0 & a^2b^2 \end{pmatrix}$$

The eigenvalues of this matrix are $(a^2b^2, p^2q^2, 0, 0)$. Therefore the concurrence is $\max\{|ab| - |pq|, |pq| - |ab|\}$.