Solutions: End-Semester Examination

Saturday, Nov 24th, 2018; Time: 9:00 am -12:00 pm; Maximum Marks: 100

Solution 1:

(a) We have

$$\tilde{v}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(t)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[ae^{i(\omega-\omega_a)t} + be^{i(\omega-\omega_b)t}\right]d\omega = \frac{1}{2\pi}\left[a\delta(\omega-\omega_a) + b\delta(\omega-\omega_b)\right]$$

Therefore, the cross-spectral density function is given by

$$\begin{aligned} W(\omega_1,\omega_2) &= \langle\tilde{v}^*(\omega_1)\tilde{v}(\omega_2)\rangle = \tilde{v}^*(\omega_1)\tilde{v}(\omega_2) \\ &= \frac{1}{(2\pi)^2}\Big[a^2\delta(\omega_1-\omega_a)\delta(\omega_2-\omega_a) + b^2\delta(\omega_1-\omega_b)\delta(\omega_2-\omega_b) \\ &\qquad + ab\delta(\omega_1-\omega_a)\delta(\omega_2-\omega_b) + ab\delta(\omega_1-\omega_b)\delta(\omega_2-\omega_a)\Big] \end{aligned}$$

(b) The intensity is given by

$$I(t) = \langle V^*(t)V(t)\rangle = V^*(t)V(t) = a^2 + b^2 + 2ab\cos(\omega_a-\omega_b)t$$

(c) the temporal correlation function is given by

$$\begin{aligned} \Gamma(t_1,t_2) &= \langle V^*(t_1)V(t_2)\rangle = V^*(t_1)V(t_2) \\ &= (ae^{i\omega_a t_1} + be^{i\omega_b t_1})(ae^{-i\omega_a t_2} + be^{-i\omega_b t_2}) \\ &= a^2e^{i\omega_a(t_1-t_2)} + b^2e^{i\omega_b(t_1-t_2)} + abe^{i(\omega_a t_1-\omega_b t_2)} + abe^{i(\omega_b t_1-\omega_a t_2)} \end{aligned}$$

(d) The degree of temporal coherence is given by

$$\gamma(t_1,t_2) = \frac{|\Gamma(t_1,t_2)|}{\sqrt{\Gamma(t_1,t_1)\Gamma(t_2,t_2)}} = \frac{|V^*(t_1)V(t_2)|}{\sqrt{V^*(t_1)V(t_1)V^*(t_2)V(t_2)}} = 1$$

Solution 2:

The conditions for the coherent-mode representation of a cross-spectral density function $W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$ are:

(i) $W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$ is square integrable, that is, $\iint_D |W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)|^2 d^2\boldsymbol{\rho}_1 d^2\boldsymbol{\rho}_2 < \infty$;

(ii) $W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$ is Hermitian, that is, $W^*(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = W(\boldsymbol{\rho}_2,\boldsymbol{\rho}_1)$; and

(iii) $W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$ is non-negative definite function, that is, $\iint_D f^*(\boldsymbol{\rho}_1)f(\boldsymbol{\rho}_2)W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)d^2\boldsymbol{\rho}_1 d^2\boldsymbol{\rho}_2 \geq 0$,

Solution 3:

(a) We have

$$U(\boldsymbol{\rho}; z) = e^{ikz} \iint_{-\infty}^{\infty} a(\boldsymbol{q}) e^{i\boldsymbol{q}\cdot\boldsymbol{\rho}} e^{-i\frac{q^2}{2k}z} d^2\boldsymbol{q}.$$

Therefore we write,

$$\langle U^*(\boldsymbol{\rho}_1; z) U(\boldsymbol{\rho}_2; z)\rangle = \iint_{-\infty}^{\infty} \langle a^*(\boldsymbol{q_1}) a(\boldsymbol{q_2})\rangle e^{-i\boldsymbol{q_1}\cdot\boldsymbol{\rho}_1 + i\boldsymbol{q_2}\cdot\boldsymbol{\rho}_2} e^{i\frac{q_1^2 - q_2^2}{2k}z} d^2\boldsymbol{q_1} d^2\boldsymbol{q_2}$$

We recognize $\langle U^*(\boldsymbol{\rho}_1; z) U(\boldsymbol{\rho}_2; z)\rangle = W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z)$ as the cross-spectral density function. The quantity $\mathcal{A}(\boldsymbol{q_1}, \boldsymbol{q_2}) \equiv \langle a^*(\boldsymbol{q_1}) a(\boldsymbol{q_2})\rangle$ is called the angular correlation function. We therefore have the cross-spectral density function in terms of the angular correlation function given as

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \iint_{-\infty}^{\infty} \mathcal{A}(\boldsymbol{q_1}, \boldsymbol{q_2}) e^{-i\boldsymbol{q_1}\cdot\boldsymbol{\rho}_1 + i\boldsymbol{q_2}\cdot\boldsymbol{\rho}_2} e^{i\frac{q_1^2 - q_2^2}{2k}z} d^2\boldsymbol{q_1} d^2\boldsymbol{q_2}.$$

(b) Since $\mathcal{A}(\boldsymbol{q}_1, \boldsymbol{q}_2) = \delta(\boldsymbol{q}_1 - \boldsymbol{q}_2)$, we have

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \int_{-\infty}^{\infty} I(\boldsymbol{q}_1) e^{-i\boldsymbol{q}_1\cdot(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)} d\boldsymbol{q_1} = \int_{-\infty}^{\infty} I(\boldsymbol{q}) e^{-i\boldsymbol{q}\cdot(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)} d\boldsymbol{q}.$$

Because of the delta-function involved, the integral contributes only when $q_1^2 = q_2^2$ and therefore, the the cross-spectral density function ends up being independent of the propagation distance z.

Solution 4:

Let us consider a unit-amplitude diverging spherical wave-field emanating from point $\boldsymbol{r_1'}$. The resulting field at position $\boldsymbol{r_1}$ due to this diverging wave can be given by

$$\frac{e^{ikR_1}}{R_1}$$

where $R_1 = |\boldsymbol{r_1} - \boldsymbol{r_1'}|$. In fact, this wave satisfies the Helmholtz equation and so can be used as a basis for representing the propagation of fields. Now, if the amplitude of the field at $\boldsymbol{r_1'}$ is $\tilde{v}(\boldsymbol{r_1'})$ then the resulting field at $\boldsymbol{r_1}$ is given by

$$\tilde{v}(\boldsymbol{r_1'})\frac{e^{ikR_1}}{R_1}.$$

Let us now consider an infinitesimal area element $d^2\boldsymbol{r_1'}$ centered at $\boldsymbol{r_1'}$. The resulting field due to the area element thus becomes

$$\tilde{v}(\boldsymbol{r_1'})\frac{e^{ikR_1}}{R_1}\Lambda_1(k) d^2\boldsymbol{r_1'},$$

where $\Lambda_1(k)$ depends on the orientation of the area element. The total field $\tilde{v}(\boldsymbol{r_1}, \omega)$ at $\boldsymbol{r_1}$ due to the source is calculated by integrating over the entire source area, that is,

$$\tilde{v}(\boldsymbol{r_1}) = \int_{\mathcal{A}} \tilde{v}(\boldsymbol{r_1'})\frac{e^{ikR_1}}{R_1}\Lambda_1(k) d^2\boldsymbol{r_1'}.$$

Similarly, the total field $\tilde{v}(\boldsymbol{r_2})$ at some other position $\boldsymbol{r_2}$ due to the same source can be written as

$$\tilde{v}(\boldsymbol{r_2}) = \int_{\mathcal{A}} \tilde{v}(\boldsymbol{r'_2}) \frac{e^{ikR_2}}{R_2} \Lambda_2(k) d^2\boldsymbol{r'_2},$$

where, as before, $R_2 = |\boldsymbol{r_2} - \boldsymbol{r'_2}|$, and $\Lambda_2(k)$ is the inclination factor corresponding to the area element $d^2\boldsymbol{r'_2}$. Using the field representations above, we obtain the following expression for the field correlation $\langle \tilde{v}^*(\boldsymbol{r_1})\tilde{v}(\boldsymbol{r_2})\rangle$:

$$\langle \tilde{v}^*(\boldsymbol{r_1})\tilde{v}(\boldsymbol{r_2})\rangle = \int_{\mathcal{A}}\int_{\mathcal{A}} \langle \tilde{v}^*(\boldsymbol{r'_1})\tilde{v}(\boldsymbol{r'_2})\rangle \frac{e^{ik(R_2-R_1)}}{R_1R_2} \Lambda_1^*(k)\Lambda_2(k) d^2\boldsymbol{r'_1} d^2\boldsymbol{r'_2}.$$

The quantities in the angular brackets are the cross-spectral densities function. So, we can rewrite the above equation as

$$W(\boldsymbol{r_1}, \boldsymbol{r_2}) = \int_{\mathcal{A}}\int_{\mathcal{A}} W(\boldsymbol{r'_1}, \boldsymbol{r'_2}) \frac{e^{ik(R_2-R_1)}}{R_1R_2} \Lambda_1^*(k)\Lambda_2(k) d^2\boldsymbol{r'_1} d^2\boldsymbol{r'_2}.$$

The above is the propagation equation for spatial coherence, which expresses the spatial correlations of the propagated field in terms of the spatial correlations at the source. It can be rigourously shown that $\Lambda_1(k) = \Lambda_2(k) = \frac{ik}{2\pi}$ but we will just use this result and will not derive it here. Now we have $W(\boldsymbol{r'_1}, \boldsymbol{r'_2}) = S(\boldsymbol{r'_1}, \omega)\delta(\boldsymbol{r'_2} - \boldsymbol{r'_1})$. Therefore the cross spectral density function at $(\boldsymbol{r_1}, \boldsymbol{r_2})$ can be written as

$$W(\boldsymbol{r_1}, \boldsymbol{r_2}) = \left(\frac{k}{2\pi}\right)^2 \int_{\mathcal{A}}\int_{\mathcal{A}} S(\boldsymbol{r'_1})\delta(\boldsymbol{r'_2} - \boldsymbol{r'_1}) \frac{e^{ik(R_2-R_1)}}{R_1R_2} d^2\boldsymbol{r'_1} d^2\boldsymbol{r'_2},$$

$$\text{or,} \quad W(\boldsymbol{r_1}, \boldsymbol{r_2}) = \left(\frac{k}{2\pi}\right)^2 \int_{\mathcal{A}} S(\boldsymbol{r'_1}) \frac{e^{ik(R_2-R_1)}}{R_1R_2} d^2\boldsymbol{r'_1},$$

In the far-zone, we can take $R_1 = r_1 - |r'_1|\cos\phi = r_1 - \boldsymbol{r'_1}.\boldsymbol{s_1}$. Similarly, we have $R_2 = r_2 - \boldsymbol{r'_1}.\boldsymbol{s_2}$ in the far zone. Also, we can safely assume that the factor $\frac{1}{R_1R_2}$ does not vary much in the far-zone for the integration-range of interest and so we can take it out of the integral. The far-zone form of the above equation is then

$$W(\boldsymbol{r_1}, \boldsymbol{r_2}) = \left(\frac{k}{2\pi}\right)^2 \frac{1}{R_1R_2} \int_{\mathcal{A}} S(\boldsymbol{r'_1}) e^{ik(r_2-r_1)} e^{-ik(\boldsymbol{s_2}-\boldsymbol{s_1}).\boldsymbol{r'_1}} d^2\boldsymbol{r'_1},$$

$$\text{or,} \quad W(\boldsymbol{r_1}, \boldsymbol{r_2}) = \left(\frac{k}{2\pi}\right)^2 \frac{e^{ik(r_2-r_1)}}{R_1R_2} \int_{\mathcal{A}} S(\boldsymbol{r'_1}) e^{-ik(\boldsymbol{s_2}-\boldsymbol{s_1}).\boldsymbol{r'_1}} d^2\boldsymbol{r'_1}$$

Solution 5:

The state of the photon can be written in the H/V basis as

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}$$

We now need to write the state in the $45/-45$ basis. The corresponding unitary operator is $\hat{U} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Therefore the state in the $45/-45$ basis can be written as

$$\rho' = \hat{U}^{-1}\rho\hat{U} = \hat{U}^\dagger\rho\hat{U}$$

Here we have used the fact that for unitary operator $\hat{U}^{-1} = \hat{U}^\dagger$. The state of the photon in the $45/-45$ basis can therefore shown to be

$$\rho' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}$$

Thus, we find that the probability of detecting the photon as a 45^0-polarized photon is 3/4.

Solution 6:

The state of the photon ρ can be written as

$$\rho = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/6 \\ 1/6 & 1/2 \end{pmatrix}$$

The degree of polarization P is given by $P = \left[1 - 4\frac{\text{Det}\rho}{(\text{Tr}\rho)^2}\right]^{1/2}$. For the above state $\text{Tr}\rho = 1$ and $\text{Det}\rho = 8/36$. Therefore we have

$$P = \left[1 - 4\frac{\text{Det}\rho}{(\text{Tr}\rho)^2}\right]^{1/2} = \left[1 - 4 \times \frac{8}{36}\right]^{1/2} = \left[\frac{4}{36}\right]^{1/2} = \frac{1}{3}$$

Solution 7:

From the definition of the degree of polarization, we know that a 2×2 polarization matrix can always be decomposed uniquely into two matrices, one of which is completely polarized and the other completely unpolarized. So, if the degree of polarization is P, then the polarization matrix J can be written as

$$J = PJ_{\text{pol}} + (1 - P)\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix},$$

where J_{pol} is the completely polarized matrix with unit trace and $\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$ is the completely unpolarized matrix with unit trace. For the given polarization matrix we have,

$$P = \left[1 - 4\frac{\text{Det}J}{(\text{Tr}J)^2}\right]^{1/2} = [1 - 4\text{Det}J]^{1/2} = \left[1 - 4\left(\frac{5}{8}\cdot\frac{3}{8} - \frac{\sqrt{3}}{8}\cdot\frac{\sqrt{3}}{8}\right)\right]^{1/2} = \left[1 - \frac{12}{16}\right]^{1/2} = \frac{1}{2}$$

Now from the above decomposition, we know that

$$PJ_{\text{pol}} = J - (1 - P)\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix},$$

$$\text{Or,} \qquad \frac{1}{2}J_{\text{pol}} = \begin{pmatrix} \frac{5}{8} & \frac{\sqrt{3}}{8} \\ \frac{\sqrt{3}}{8} & \frac{3}{8} \end{pmatrix} - \begin{pmatrix} \frac{2}{8} & 0 \\ 0 & \frac{2}{8} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{8} & \frac{\sqrt{3}}{8} \\ \frac{\sqrt{3}}{8} & \frac{1}{8} \end{pmatrix}$$

Therefore, we have

$$J_{\text{pol}} = \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}$$

And thus J can be written uniquely as

$$J = \frac{1}{2}\begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} \frac{3}{8} & \frac{\sqrt{3}}{8} \\ \frac{\sqrt{3}}{8} & \frac{1}{8} \end{pmatrix} + \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix},$$

where the first matrix is completely polarized and the second one is completely unpolarized.

Solution 8:

(a) $\alpha = \beta^*$

(b) We have $\alpha = |\alpha|e^{i\theta}$, where $0 \leq \theta \leq 2\pi$, and $0 \leq |\alpha| \leq \sqrt{0.64 \times 0.36}$, that is, $0 \leq |\alpha| \leq 0.48$

Solution 9:

The only instances when Alice and Bob will keep the measurement result are when they both happen to use the same base. And in these instances Alice and Bob will obtain the same measurement results since there is no eavesdropping or channel errors. Therefore the generated key that Alice and Bob will share after the protocol is : 101011001.

Solution 10:

Since $\gg d \gg l_{\rm coh}$ and the coincidence time-window is much smaller compared to the time $(x_0 - d)/c$, we find that there can be only two possible ways in which the signal an idler photons can reach the detectors in coincidence. One in which both the photons travel the shorter paths and the second in which both the photons travel the longer paths. Since we are assuming no loss, perfect mirror and 50:50 beam splitters, the coincidence count rate of the signal and idler detector is given by

$$R_{si} = C\left[1 + \gamma(\Delta L)\gamma(\Delta L')\cos\left(\frac{2\pi}{\lambda_0}\Delta L + \Delta\phi\right)\right]$$

For the given interferometer, it can be shown that

$$\begin{aligned}
&\Delta L = x_s + x_i + 2x_0 - 2d \\
&\Delta L' = 2(x_s - x_i) \\
&\Delta\phi = 0
\end{aligned}$$

Since the pump is monochromatic, we have $\gamma(\Delta L) = 1$, and therefore the coincidence count rate can be written as

$$R_{si} = C\left[1 + \gamma(\Delta L')\cos\left(\frac{2\pi}{\lambda_0}\Delta L\right)\right]$$

(a) In order to see a HOM-like dip profile, one has to make sure that one changes only $\Delta L'$ while keeping ΔL fixed. Given the expressions for ΔL and $\Delta L'$ above, we find that this can be done if the signal and idler mirrors and displaced by equal amounts in opposite directions, that is,

$$\begin{aligned}
x_s &= a + x \\
x_i &= -x
\end{aligned}$$

where a is some fixed starting distance and x is the variable distance. Now, the other condition that must be satisfied for the observance of a dip is that $\cos\left(\frac{2\pi}{\lambda_0}\Delta L\right) = -1$. This requires that

$$\begin{aligned}
&\Delta L = (2m+1)\pi \times \frac{\lambda_0}{2\pi} = (2m+1)\frac{\lambda_0}{2}, \quad \text{where} \quad m = 0, 1, 2, \cdots \\
\text{Or,} \quad & x_s + x_i + 2x_0 - 2d = (2m+1)\frac{\lambda_0}{2} \\
\text{Or,} \quad & a + x - x + 2x_0 - 2d = (2m+1)\frac{\lambda_0}{2} \\
\text{Or,} \quad & a = (2m+1)\frac{\lambda_0}{2} - 2x_0 + 2d
\end{aligned}$$

So, in order to observe a HOM-like dip profile, the signal mirror should first be displaced from the balanced position by an amount $a = (2m+1)\frac{\lambda_0}{2} - 2x_0 + 2d$, and then both the signal and idler mirrors should be displaced in opposite directions by equal amount x. Since $\Delta L' = 2(x_s - x_i) = 4x + 4a$, we would observe the dip as a function of x, and it would be centered at $x = -\frac{a}{2}$.

(b) In order to see a HOM-like hump profile, one has to make sure that one changes only $\Delta L'$ while keeping ΔL fixed. Given the expressions for ΔL and $\Delta L'$ above, we find that this can be done if the signal and idler mirrors and displaced by equal amounts in opposite directions, that is,

$$\begin{aligned} x_s &= a + x \\ x_i &= -x \end{aligned}$$

where a is some fixed starting distance and x is the variable distance. Now, the other condition that must be satisfied for the observance of a hump is that $\cos\left(\frac{2\pi}{\lambda_0}\Delta L\right) = 1$. This requires that

$$\begin{aligned} & \Delta L = 2m\pi \times \frac{\lambda_0}{2\pi} = m\lambda_0, \quad \text{where} \quad m = 0, 1, 2, \cdots \\ \text{Or,} \quad & x_s + x_i + 2x_0 - 2d = m\lambda_0 \\ \text{Or,} \quad & a + x - x + 2x_0 - 2d = m\lambda_0 \\ \text{Or,} \quad & a = m\lambda_0 - 2x_0 + 2d \end{aligned}$$

So, in order to observe a HOM-like hump profile, the signal mirror should first be displaced from the balanced position by an amount $a = m\lambda_0 - 2x_0 + 2d$, and then both the signal and idler mirrors should be displaced in opposite directions by equal amount x. Since $\Delta L' = 2(x_s - x_i) = 4x + 4a$, we would observe the hump as a function of x, and it would be centered at $x = -\frac{a}{2}$.

(c) Since the two entangled photons in this setup never come together, there can be no bunching interpretation in this interferomter of the HOM-like dip profiles.

Solution 11: We note that the combined state of the two entangled particles and particle-C can be written as a pure tensor product of $|\Psi^-\rangle_{\rm AB}$ and $|\phi\rangle_{\rm C}$ and is given as

$$\begin{aligned} |\Psi\rangle_{\rm ABC} &= |\Psi^-\rangle_{\rm AB}|\phi\rangle_{\rm C} \\ &= \frac{1}{\sqrt{2}}\left(|H\rangle_{\rm A}|V\rangle_{\rm B} - |V\rangle_{\rm A}|H\rangle_{\rm B}\right)\left(\alpha|H\rangle_{\rm C} + \beta|V\rangle_{\rm C}\right) \\ &= \frac{\alpha}{\sqrt{2}}|H\rangle_{\rm A}|H\rangle_{\rm C}|V\rangle_{\rm B} - \frac{\alpha}{\sqrt{2}}|V\rangle_{\rm A}|H\rangle_{\rm C}|H\rangle_{\rm B} + \frac{\beta}{\sqrt{2}}|H\rangle_{\rm A}|V\rangle_{\rm C}|V\rangle_{\rm B} - \frac{\beta}{\sqrt{2}}|V\rangle_{\rm A}|V\rangle_{\rm C}|H\rangle_{\rm B}. \end{aligned} \tag{1}$$

Alice has two particles, particle-A and particle-C. She performs a Bell state-measurement on these particle. This simply means that she wants to see which Bell state her two particles are in. The Bell-basis representing particle-A and particle-C are given as

$$\begin{aligned} |\Phi^\pm\rangle_{\rm AC} &= \frac{1}{\sqrt{2}}\left[|H\rangle_{\rm A}|H\rangle_{\rm C} \pm |V\rangle_{\rm A}|V\rangle_{\rm C}\right], \\ |\Psi^\pm\rangle_{\rm AC} &= \frac{1}{\sqrt{2}}\left[|H\rangle_{\rm A}|V\rangle_{\rm C} \pm |V\rangle_{\rm A}|H\rangle_{\rm C}\right]. \end{aligned}$$

The above relations yield

$$\begin{aligned} |H\rangle_{\rm A}|H\rangle_{\rm C} &= \frac{1}{\sqrt{2}}\left(|\Phi^+\rangle_{\rm AC} + |\Phi^-\rangle_{\rm AC}\right), \\ |V\rangle_{\rm A}|H\rangle_{\rm C} &= \frac{1}{\sqrt{2}}\left(|\Psi^+\rangle_{\rm AC} - |\Psi^-\rangle_{\rm AC}\right), \\ |H\rangle_{\rm A}|V\rangle_{\rm C} &= \frac{1}{\sqrt{2}}\left(|\Psi^+\rangle_{\rm AC} + |\Psi^-\rangle_{\rm AC}\right), \\ |V\rangle_{\rm A}|V\rangle_{\rm C} &= \frac{1}{\sqrt{2}}\left(|\Phi^+\rangle_{\rm AC} - |\Phi^-\rangle_{\rm AC}\right). \end{aligned}$$

Using the above relations Eq. (**??**) can be written as

$$|\Psi\rangle_{\rm ABC} = \frac{\alpha}{2}\left(|\Phi^{+}\rangle_{\rm AC} + |\Phi^{-}\rangle_{\rm AC}\right)|V\rangle_{\rm B} - \frac{\alpha}{2}\left(|\Psi^{+}\rangle_{\rm AC} - |\Psi^{-}\rangle_{\rm AC}\right)|H\rangle_{\rm B} + \frac{\beta}{2}\left(|\Psi^{+}\rangle_{\rm AC} + |\Psi^{-}\rangle_{\rm AC}\right)|V\rangle_{\rm B} - \frac{\beta}{2}\left(|\Phi^{+}\rangle_{\rm AC} - |\Phi^{-}\rangle_{\rm AC}\right)|H\rangle_{\rm B}. \quad (2)$$

The above equation can be rewritten as

$$|\Psi\rangle_{\rm ABC} = \frac{1}{2}|\Phi^{+}\rangle_{\rm AC}\left(\alpha|V\rangle_{\rm B} - \beta|H\rangle_{\rm B}\right) + \frac{1}{2}|\Phi^{-}\rangle_{\rm AC}\left(\alpha|V\rangle_{\rm B} + \beta|H\rangle_{\rm B}\right) + \frac{1}{2}|\Psi^{+}\rangle_{\rm AC}\left(-\alpha|H\rangle_{\rm B} + \beta|V\rangle_{\rm B}\right) + \frac{1}{2}|\Psi^{-}\rangle_{\rm AC}\left(\alpha|H\rangle_{\rm B} + \beta|V\rangle_{\rm B}\right) \quad (3)$$

We find that particle-A and particle-C are equally likely to be found in any of the four Bell states and the probability is 1/4. So, after Alice has done the Bell-state analysis she sends that information to Bob and based on that information Bob deicides on a unitary transformation which guarantees that he has the original state. Here is how it is done:

(1) Alice's measurement gives her the Bell state $|\Phi^{+}\rangle_{\rm AC}$. The state of Bob's qubit will be $(\alpha|V\rangle_{\rm B} - \beta|H\rangle_{\rm B})$. So, Bob makes the unitary transformation given by $\hat{U}_{\phi^+} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ such that the state of Bob's qubit becomes

$$U_{\phi^+}\left(\alpha|V\rangle_{\rm B} - \beta|H\rangle_{\rm B}\right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -\beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \left(\alpha|H\rangle_{\rm B} + \beta|V\rangle_{\rm B}\right) \quad (4)$$

(2) Alice's measurement gives her the Bell state $|\Phi^{-}\rangle_{\rm AC}$. The state of Bob's qubit will be $(\alpha|V\rangle_{\rm B} + \beta|H\rangle_{\rm B})$. So, Bob makes the unitary transformation given by $\hat{U}_{\phi^-} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ such that the state of Bob's qubit becomes

$$U_{\phi^-}\left(\alpha|V\rangle_{\rm B} + \beta|H\rangle_{\rm B}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \left(\alpha|H\rangle_{\rm B} + \beta|V\rangle_{\rm B}\right) \quad (5)$$

(3) Alice's measurement gives her the Bell state $|\Psi^{+}\rangle_{\rm AC}$. The state of Bob's qubit will be $(-\alpha|H\rangle_{\rm B} + \beta|V\rangle_{\rm B})$. So, Bob makes the unitary transformation given by $\hat{U}_{\psi^+} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ such that the state of Bob's qubit becomes

$$U_{\psi^+}\left(-\alpha|H\rangle_{\rm B} + \beta|V\rangle_{\rm B}\right) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \left(\alpha|H\rangle_{\rm B} + \beta|V\rangle_{\rm B}\right) \quad (6)$$

(4) Alice's measurement gives her the Bell state $|\Psi^{-}\rangle_{\rm AC}$. The state of Bob's qubit will be $(\alpha|H\rangle_{\rm B} + \beta|V\rangle_{\rm B})$. So, Bob makes the unitary transformation given by $\hat{U}_{\psi^-} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ such that the state of Bob's qubit becomes

$$U_{\psi^-}\left(\alpha|H\rangle_{\rm B} + \beta|V\rangle_{\rm B}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \left(\alpha|H\rangle_{\rm B} + \beta|V\rangle_{\rm B}\right) \quad (7)$$

Thus, we see that Bob gets the same state $|\psi\rangle$ that Alice intended to teleport.