Department of Physics
IIT Kanpur, Semester II, 2016-17

PSO201A: Quantum Physics — **Homework # 4**

Problem 4.1: Taking the adjoint

Write down the adjoint of the following equations/expressions

(a) $\hat{\Omega}|V_1\rangle + \sum_{i=0}^{n} v_i|i\rangle$

(b) $|\psi_1\rangle = a|\psi_2\rangle + b\langle\psi_3|\psi_4\rangle|\psi_5\rangle + d\langle\psi_6|\hat{\Omega}|\psi_7\rangle|\psi_8\rangle$

Problem 4.2: Representation in position basis

Write down the following vector-equations/-expressions in the position basis.

(a) $\langle\psi_1|\psi_2\rangle$

(b) $\langle x|x'\rangle$

(c) $\langle\psi_1|\hat{\Omega}|\psi_2\rangle$

(d) $\hat{\Omega}|\psi_1\rangle = \omega|\psi_1\rangle$

(e) $\langle\psi_1|\hat{\Omega}|\psi_2\rangle = \langle\psi_2|\hat{\Omega}^\dagger|\psi_1\rangle^*$

Problem 4.3: Miscellaneous Concepts

(a) Do the following three vectors form a basis:

$$|1\rangle = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \qquad |2\rangle = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}; \qquad |3\rangle = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}?$$

(b) Show that the unitary transformations preserve the inner product. That is, show that if $\hat{\mathrm{U}}$ is a unitary operator and $|\psi_1\rangle$ and $|\psi_2\rangle$ are two vectors then $\langle\psi_1'|\psi_2'\rangle = \langle\psi_1|\psi_2\rangle$, where $|\psi_1'\rangle = \hat{\mathrm{U}}|\psi_1\rangle$ and $|\psi_2'\rangle = \hat{\mathrm{U}}|\psi_2\rangle$.

(c) The matrix representations for $\hat{\mathrm{X}}$ and $\hat{\mathrm{P}}$ in the position basis are given by: $\langle x|\hat{\mathrm{X}}|x'\rangle = \delta(x-x')x'$ and $\langle x|\hat{\mathrm{P}}|x'\rangle = \delta(x-x')\left(-i\hbar\frac{d}{dx'}\right)$. Show that the matrix representation for an operator $f(\hat{\mathrm{X}},\hat{\mathrm{P}})$ that is a function of $\hat{\mathrm{X}}$ and $\hat{\mathrm{P}}$, can be written as

$$\langle x|f(\hat{\mathrm{X}},\hat{\mathrm{P}})|x'\rangle = \delta(x-x')f\left(x', -i\hbar\frac{d}{dx'}\right).$$

(d) Show that the momentum operator $\hat{\mathrm{P}}$, with the position-basis representation given by $\langle x|\hat{\mathrm{P}}|x'\rangle = \delta(x-x')\left(-i\hbar\frac{d}{dx'}\right)$, is a Hermitian operator.

(e) If $\hat{\mathrm{H}}$ is a Hermitian matrix, then show that $e^{i\hat{\mathrm{H}}}$ is a unitary matrix.

(f) Show that the eigenvectors of a Hermitian operator are mutually orthogonal (assume that there is no degeneracy).

(g) If $\hat{\text{A}}$ and $\hat{\text{B}}$ are two Hermitian operators then under what condition is the product $\hat{\text{A}}\hat{\text{B}}$ also a Hermitian operator?

(h) Show that the commutator of position and momentum operators $[\hat{\text{X}}, \hat{\text{P}}] = i\hbar\hat{I}$, where $\hat{I}$ is the identity matrix. (Hint: show that $[\hat{\text{X}}, \hat{\text{P}}]|\psi\rangle = i\hbar\hat{I}|\psi\rangle$, where $|\psi\rangle$ is any vector in the Hilbert space.)

Problem 4.4: Diagonalizing a Matrix

(a) For the Unitary matrix $\hat{A}$ below, show that the eigenvalues are complex number of unit modulus and that the eigenvectors are orthogonal.

$$\hat{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

(b) Verify that $\hat{U}^\dagger \hat{A} \hat{U}$ is a diagonal matrix, where $\hat{U}$ is the matrix of the eigenvectors of $\hat{A}$.

Problem 4.5: Completeness of a basis

(a) Show that the following three vectors:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \qquad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

form a complete basis.

(b) Let $|p\rangle$ be the eigenvector of the momentum operator. The completeness condition for the momentum eigenvectors is given by $\int_{-\infty}^{\infty} |p\rangle\langle p| dp = \text{I}$, where I is the identity matrix. Show that the completeness condition for the momentum eigenvectors can be written in the position basis as

$$\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{ip(x-x')/\hbar} dp = \delta(x - x')$$

(c) Let $\psi_n(x)$ is the position-space eigenfunction of the simple harmonic oscillator Hamiltonian. Evaluate the following sum:

$$\sum_{n=0}^{\infty} \psi_n(x)\psi_n^*(x'),$$

$$\text{where} \quad \psi_n(x) = A_n \exp\left(-\frac{m\omega x^2}{2\hbar}\right) H_n\left[\left(\frac{m\omega}{\hbar}\right)^{1/2} x\right]$$

(Hint: Use the fact that the eigenvectors of a Hermitian operator form a complete basis.)