Department of Physics
IIT Kanpur, Semester II, 2016-17

PSO201A: Quantum Physics **Homework # 5**

Problem 5.1: Uncertainty Principle

(a) Let us consider a bullet of mass 60 g and an electron of mass (9.1×10^{-31}) Kg, each moving with speed 200 m/s. If one could determine the speeds of both the electron and the bullet to within an accuracy of 0.01%, how accurately could one measure the positions of the electron and the bullet.

(b) Comment on how the uncertainly principle affects the accuracy of position measurements for microscopic versus macroscopic objects.

(c) After going to an excited state, an atom emits a photon and comes back to the ground state. Suppose there is an uncertainly of about one nanosecond as to when precisely the atom emits the photon. What is the uncertainty in the energy of the emitted photons?

(d) A pulse laser emits out pulses of light at regular interval. Suppose the laser is emitting out pulses of one nanosecond duration with a mean frequency ν_0. What is the frequency content of such a laser.

Problem 5.2: Some Conceptual Questions

(a) If $\psi(x,t)$ represents a wave-function, what is the meaning of $|\psi(x,t)|^2 = \psi(x,t)^*\psi(x,t)$?

(b) Mathematically, continuity and square-integrability are the two most important conditions that a wave-function has to satisfy. Explain on the physical grounds the origin of these two conditions?

(c) Show that if $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are the two solutions to the Schrödinger equation then $\Psi(x,t) = A\Psi_1(x,t) + B\Psi_2(x,t)$ is a also a solution to the Schrödinger equation.

(d) If $\psi_n(x)$ is a solution to the time-independent Schrödinger equation, with energy E_n, show that $\Psi(x,t) = \sum_n A_n\psi_n(x)e^{-iE_nt/\hbar}$ is a solution to the time-dependent Schrödinger equation.

(e) Let $\psi_n(x)$ be a solution to the time-independent Schrödinger equation, with energy E_n. Show that $\Psi(x,t) = \sum_n A_n\psi_n(x)e^{-iE_nt/\hbar}$ is not a stationary solution to the Schrödinger equation but $\Psi_n(x,t) = [A\psi_n(x) + B\psi_n^*(x)]e^{-iE_nt/\hbar}$ is a stationary solution.

(f) Can the following functions be solutions to the time-independent Schrödinger equation: (i) $\psi(x) = Ax^2$, (ii) $\psi(x) = A$, and (iii) $\psi(x) = \frac{A}{\cos x}$, (iv) $\psi(x) = Ae^{-x^2}$?

(g) One of the most often encountered quantum mechanical wave-functions is the Gaussian wave-function and is given as. $\psi(x) = A\exp\left[-\frac{(x-\mu)^2}{4\sigma^2}\right]$. Find the normalization constant A and the expectation values $\langle x\rangle$ and $\langle x^2\rangle$. Also, calculate the uncertainty $\Delta x = \sqrt{\langle(x-\langle x\rangle)^2\rangle}$

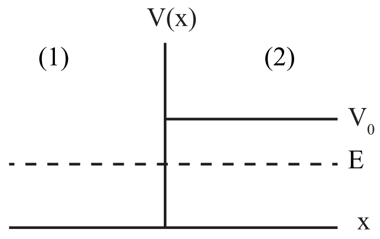
Problem 5.3: Particle in a step potential ($V_0 > E$)

Consider a particle of mass m moving with energy E in a step potential, with $V_0 > E$, as shown in the figure. The solution to the Schrödinger equation for the step potential is given as

$$\begin{aligned} \psi_1(x) &= Ae^{ik_1x} + Be^{-ik_1x} \qquad \text{when } x < 0 \\ \psi_2(x) &= De^{-k_2x} \qquad \text{when } x > 0, \end{aligned} \tag{1}$$

where $\psi_1(x)$ and $\psi_2(x)$ are the solutions in region 1 and region 2, respectively. $k_1 = \frac{\sqrt{2mE}}{\hbar}$ and $k_2 = \frac{\sqrt{2m(V_0-E)}}{\hbar}$; and $A = \frac{D}{2}\left(1 + \frac{ik_2}{k_1}\right)$, and $B = \frac{D}{2}\left(1 - \frac{ik_2}{k_1}\right)$.

(a) Calculate the probability densities in both the regions.



(b) Is the above solution a stationary solution?

(c) Assume that the particle is an electron with energy $E = 1$ eV and take $V_0 = 5$ eV and $|D|^2 = 1$. Plot the probability densities in the range $-2\lambda_1 < x < 2\lambda_1$, where λ_1 is the de-Broglie wavelength in region 1.

(d) What is the penetration depth of the electron in region 2?

(e) Next, assume that the particle is an electron with energy $E = 1$ eV and take $V_0 = 1.25$ eV and $|D|^2 = 1$. Plot the probability densities in the range $-2\lambda_1 < x < 2\lambda_1$, where λ_1 is the de-Broglie wavelength in region 1.

(f) What is the penetration depth of the electron in region 2?

(g) The probability density in region 1 is an oscillating function. Explain if this is similar to the interference pattern observed in a Young's double-slit experiment.

(h) Can such interference pattern in probability density be seen with a classical particle of macroscopic mass and momentum? Explain qualitatively.

Problem 5.4: Particle in a step potential ($V_0 < E$)

Consider a particle of mass m moving with energy E in a step potential, with $V_0 < E$, as shown in the figure. The solution to the step potential is given as

$$\begin{aligned} \psi_1(x) &= Ae^{ik_1x} + Be^{-ik_1x} \qquad \text{when } \ x < 0 \\ \psi_2(x) &= Ce^{ik_2x} \qquad \text{when } \ x > 0, \end{aligned} \tag{2}$$

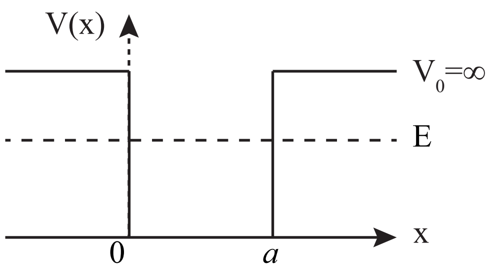
where $k_1 = \dfrac{\sqrt{2mE}}{\hbar}$ and $k_2 = \dfrac{\sqrt{2m(E-V_0)}}{\hbar}$; $B = A\left(\dfrac{k_1-k_2}{k_1+k_2}\right)$ and $C = A\left(\dfrac{2k_1}{k_1+k_2}\right)$

(a) Calculate the probability densities in both the regions.

(b) Assume that the particle is an electron with energy $E = 1$ eV and take $V_0 = 0.75$ eV and $|A|^2 = 1$. Plot the probability densities in the range $-2\lambda_1 < x < 2\lambda_1$, where λ_1 is the de-Broglie wavelength in region 1.

(c) The probability density in region 1 in this case is different from the probability density in region 1 in Problem 5.3. Explain qualitatively the reason for this difference.

Problem 5.5: Infinite-Well Potential



(a) Consider the potential shown above. Find the stationary-state solutions and the corresponding allowed energies.

(b) Calculate the uncertainty product $\Delta x \Delta p$ for the n^{th} stationary state.