Department of Physics
IIT Kanpur, Semester II, 2016-17

PSO201A: Quantum Physics

Homework # 6

Problem 6.1: Minimum uncertainty wave-function

(a) The position-space wave function for a particle in the lowest-energy stationary state of a harmonic potential is given by $\psi_0(x) = (\frac{1}{2\pi\sigma_x^2})^{1/4}\exp(-\frac{x^2}{4\sigma_x^2})$, where $\sigma_x = \sqrt{\hbar/(2m\omega)}$. Find the momentum-space wave function for the particle. (Hint: you could make use of the standard integral: $\int_{-\infty}^{\infty} e^{-\alpha x^2+\beta x}dx = \exp\left(\frac{\beta^2}{4\alpha}\right)\left(\frac{\pi}{\alpha}\right)^{1/2}$.

(b) Show that the above wave function is a minimum uncertainty wave function.

Problem 6.2: Stationary-States of Simple harmonic oscillator

(a) For the simple harmonic oscillator potential, the first two stationary state solutions are:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}x^2\right]; \qquad \psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}x\exp\left[-\frac{m\omega}{2\hbar}x^2\right]$$

Calculate the uncertainty product $\Delta x\Delta p$ for both $\psi_0(x)$ and $\psi_1(x)$.

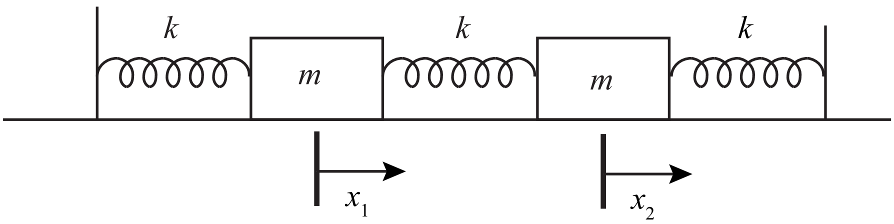
(b) For both the states, calculate the expectation values for the kinetic and the potential energy and show that the sum is equal to the expectation value for the total energy of the state.

Problem 6.3: Time-evolution of a state

(a) A particle in the harmonic oscillator potential is in the following superposition state at $t = 0$: $\psi(x, 0) = \frac{3}{5}\psi_0(x) + \frac{4}{5}\psi_1(x)$, where $\psi_0(x)$ and $\psi_1(x)$ are the stationary state solutions for the harmonic potential, as in the previous problem. Find the probability density $|\psi(x, t)|^2$ at some later time t.

(b) what is the expectation value for the energy of the system?

Problem 6.4: Coupled-mass system



(a) Consider the coupled mass system as shown in the figure. All the masses are m and the spring constants are k. The displacement of the two masses from the equilibrium positions are given by x_1 and x_2, respectively. Find the potential energy of the system.

(b) Write down the time-independent Schrödinger equation for the system.

(c) Break the above Schrödinger equation representing a coupled-mass system into two independent Schrödinger equations. (Hint: Find an appropriate coordinate transformation and use the separation of variables to obtain two uncoupled equations in terms of the new coordinates.)

(d) What are the allowed energy values that the system can have?

(e) What are the corresponding stationary state solutions that the system can have?

Problem 6.5: Half Harmonic potential

Consider a particle of mass m in a 'half' harmonic potential given by:

$$V(x) = \begin{cases} (1/2)m\omega^2 x^2 & \text{if } x > 0, \\ \infty & \text{if } x < 0, \end{cases}$$

Find the allowed energies and the corresponding wave-function solutions for the particle.

Problem 6.6: Infinite-Well potential dynamics - I

Consider a particle of mass m in an infinite square-well potential of width $2a$ centered at $x = 0$. Suppose that the particle is in its first stationary state given by

$$\Psi_1(x,t) = \begin{cases} \sqrt{\frac{1}{a}}\cos(k_1 x)e^{-iE_1 t/\hbar} & \text{if } -a < x < a, \\ 0 & \text{otherwise,} \end{cases}$$

$k_1 = \dfrac{\pi}{2a}$ and $E_1 = \dfrac{\pi^2\hbar^2}{8ma^2}$.

(a) Calculate the expectation value for position $\langle x(t)\rangle$ as a function of time.

(b) Calculate the expectation value for momentum $\langle p(t)\rangle$ as a function of time.

(c) Calculate the uncertainty in the position measurement $\Delta x(t)$.

(d) Calculate the uncertainty in the momentum measurement $\Delta p(t)$.

(e) Verify that the uncertainty product $\Delta x(t)\Delta p(t)$ obeys the Heisenberg uncertainty relation.

Problem 6.7: Infinite-Well potential dynamics - II

Consider again a particle in an infinite square-well potential of width $2a$ (as shown in the figure above). The particle is in the superposition of the first two normalized stationary states, given by $\Psi(x,t) = \frac{1}{\sqrt{2}}[\Psi_1(x,t) + \Psi_2(x,t)]$ with

$$\Psi_1(x,t) = \begin{cases} \frac{1}{\sqrt{a}}\cos(k_1 x)e^{-iE_1 t/\hbar} & \text{if } -a < x < a, \\ 0 & \text{otherwise,} \end{cases}$$

$$\Psi_2(x,t) = \begin{cases} \frac{1}{\sqrt{a}}\sin(k_2 x)e^{-iE_2 t/\hbar} & \text{if } -a < x < a, \\ 0 & \text{otherwise,} \end{cases}$$

where $k_1 = \dfrac{\pi}{2a}$ and $E_1 = \dfrac{\pi^2\hbar^2}{8ma^2}$, and $k_2 = \dfrac{\pi}{a}$ and $E_2 = \dfrac{\pi^2\hbar^2}{2ma^2}$.

(a) Verify that the above state $\Psi(x,t)$ is normalized.

(b) Calculate the expectation value for position $\langle x(t)\rangle$ as a function of time.

(c) Calculate the expectation value for energy $\langle E(t)\rangle$ as a function of time.