Department of Physics
IIT Kanpur, Semester II, 2016-17

PSO201A: Quantum Physics # Homework # 7

Problem 7.1: Two-dimensional infinite well

(a) Find the stationary state solutions and the corresponding energy values that a particle of mass m can have in a two-dimensional infinite square potential-well of side a each, given by

$$\begin{aligned} V(x,y) =& 0 \quad \text{if} \quad 0 < x < a \quad \text{and} \quad 0 < y < a \\ =& \infty \quad \text{otherwise} \end{aligned}$$

(b) What is the energy of the ground state of such a system?

(c) What is the next higher energy value that the system can take and what are the position-space wavefunctions of the states corresponding to this energy?

Problem 7.2: Propagator Basics - I

The stationary state solutions for the simple harmonic oscillator potential at time $t = 0$ is denoted by $|\psi_n\rangle$, with the energy of the n^{th} state given by $E_n = (n + \frac{1}{2})\hbar\omega$.

(a) Using the propagator algebra find out the state $|\psi(t)\rangle$ of the particle at time t if the state of the particle at $t = 0$ is given by

$$|\psi\rangle = |\psi_m\rangle$$

(b) Using the propagator algebra find out the position-space wave-function $\psi(x,t)$ of the particle at time t, if the state of the particle at time t_0 is given by

$$|\psi(t_0)\rangle = \frac{1}{\sqrt{2}}[|\psi_2(t_0)\rangle + |\psi_3(t_0)\rangle]$$

(c) Show by explicit calculation that the propagator for the simple harmonic oscillator is a unitary opertor. That is, show that $\hat{U}(t,t_0)\hat{U}^\dagger(t,t_0) = I$, where I is the identity matrix.

Problem 7.3: Propagator Basics -II

Suppose that $|\psi_n\rangle$ are the stationary state solutions at $t = 0$ to the Schrödinger equation for some time-independent potential and E_n are the corresponding energies. Also, suppose that the propagation equation for a state of such a system can be written as $|\psi(t)\rangle = \hat{U}(t,0)|\psi(0)\rangle$, where $|\psi(0)\rangle$ and $|\psi(t)\rangle$ are the states of the system at $t = 0$ and at time t, respectively, and $\hat{U}(t,0)$ is the propagator.

(a) Assuming that $|\psi_n\rangle$ form a complete basis, derive an expression for the propagator $\hat{U}(t,0)$ in terms of $|\psi_n\rangle$.

(b) Write the propagator $\hat{U}(t,0)$ in the position basis.

(c) Write the propagator equation $|\psi(t)\rangle = \hat{U}(t,0)|\psi(0)\rangle$ in the position basis.

Problem 7.4: Two-level system

Let $\hat{H} = \begin{pmatrix} \hbar\omega & 0 \\ 0 & 2\hbar\omega \end{pmatrix}$ be the Hamiltonian and $\hat{A} = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}$ be an observable of a two-level system. Let $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ be the state of this system at $t = 0$.

(a) Calculate the normalized eigenvectors and the eigenvalues of the Hamiltonian.

(b) Calculate the expectation value of energy and of the observable $\hat{A}$ at $t = 0$.

(c) Find the 2×2 matrix representation for the propagator $\hat{U}(t, 0)$.

(d) Calculate the expectation value of energy and of the observable $\hat{A}$ at time t.

Problem 7.5: Young's double-slit type effect in time

Suppose that the position-space wavefunction of a particle at $t = 0$ is

$$\begin{aligned} \psi(x,0) &= A; \quad \text{if} \quad -\left(\frac{R+d}{2}\right) < x < -\left(\frac{R-d}{2}\right) \quad \text{and} \quad \left(\frac{R-d}{2}\right) < x < \left(\frac{R+d}{2}\right) \\ &= 0; \quad \text{otherwise.} \end{aligned}$$

(Take $R >> 2d$)

(a) Find the normalization constant A and the mean velocity with which the wave-function is moving at $t = 0$.

(b) Plot the position probability density $P(x, 0) \equiv \psi^*(x, 0)\psi(x, 0)$ at $t = 0$.

(c) Find the momentum probability density $P(p, 0) \equiv \psi^*(p, 0)\psi(p, 0)$ at $t = 0$.

(d) Using the free-particle propagator $U(x, t; x', 0) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \exp\left[-\frac{im(x-x')^2}{2\hbar t}\right]$, show that the propagator equation in the long-time limit, that is, $t \gg \frac{mR^2}{2\hbar}$, can be written as

$$\psi(x,t) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \exp\left[-\frac{imx^2}{2\hbar t}\right] \int_{-\infty}^{\infty} \exp\left[\frac{imx}{\hbar t}x'\right] \psi(x', 0)dx',$$

(e) Calculate the long-time limit position probability density $P(x, t)$.

(f) Comment on the connection between the momentum probability density $P(p, 0)$ at $t = 0$ and the long-time limit position probability density $P(x, t)$.

(g) Plot the long-time limit position probability density $P(x, t)$ as a function of x, indicating all the essential features.