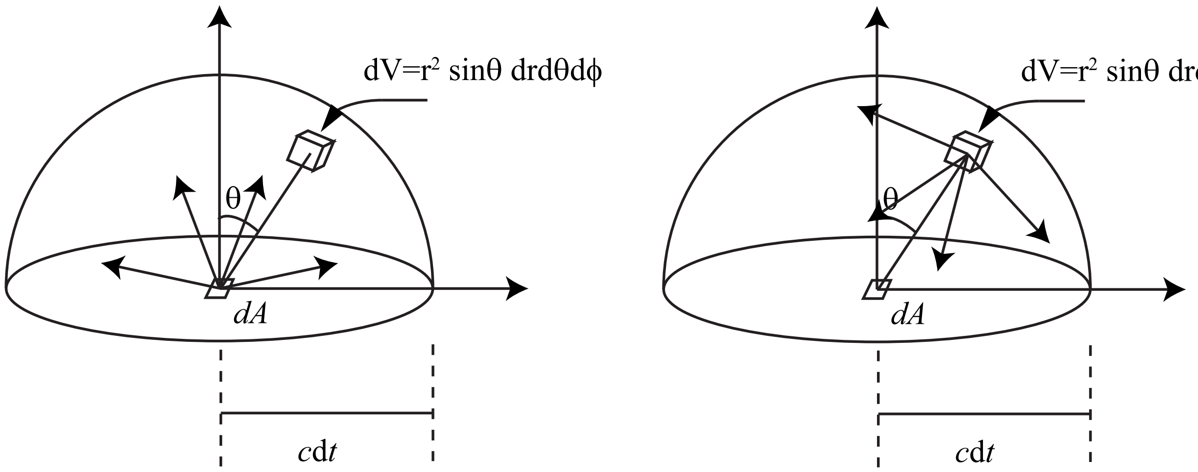
Department of Physics, IIT Kanpur, Semester II, 2016-17

PSO201A: Quantum Physics **Quiz # 1: Solutions** **Time: 35 Minutes** **Max Marks: 100**

Name: .... **Roll No.:** .... **Section**..........

Problem 1: Consider a cavity with metallic walls in thermal equilibrium as a blackbody. Find the relationship between the radiance of the blackbody and the energy density in the cavity. **(25 marks)**

Answer: In thermal equilibrium the total energy radiated out by an area element dA of a blackbody equals the energy incident on the element from the cavity (as illustrated in the figure below).



The energy E radiated out by an area element dA in time dt is

$$E = R(\nu)d\nu dAdt \tag{1}$$

This area element will receive energy from the hemispherical volume of radius cdt that is surrounding the area element. The radiation coming in from outside this region will not be able to reach the area element within dt. As shown in the figure, let's consider a volume element $dV = r^2 \sin\theta dr d\theta d\phi$ within the hemishpere. The total energy within this volume element is $\rho(\nu)d\nu dV = \rho(\nu)d\nu r^2 \sin\theta dr d\theta d\phi$. The energy dE falling from this volume element onto the area element dA which is at distance r and angle θ is $dE = \rho(\nu)d\nu r^2 \sin\theta dr d\theta d\phi \times \frac{dA}{4\pi r^2}\cos\theta$. Therefore the total energy falling onto the area element dA is

$$E = \rho(\nu)d\nu\frac{dA}{4\pi}\int_0^{cdt} dr \int_0^{\pi/2} \sin\theta\cos\theta d\theta \int_0^{2\pi} d\phi = E = \rho(\nu)d\nu dAdt\frac{c}{4} \tag{2}$$

Comparing the above two equations, we get the required relationship: $R(\nu) = \frac{c}{4}\rho(\nu)$.

Problem 2: Using Planck's hypothesis and Boltzmann's probability distribution, calculate the average energy of a standing wave at frequency ν inside a cavity at temperature T. [useful formula: $\sum_{n=0}^{\infty} x^n = 1/(1-x)$]. **(25 marks)**

Answer: Planck's hypothesis is that the standing wave inside a cavity can have energy only in the multiples of $h\nu$. With this form for the energy, and using the Boltzmann probability distribution, we get the following formula for the average energy of a standing wave at frequency ν inside a cavity at temperature T.

$$\bar{\mathcal{E}} = \frac{\sum_{n=0}^{\infty} \mathcal{E} P(\mathcal{E})}{\sum_{n=0}^{\infty} P(\mathcal{E})} = \frac{\sum_{n=0}^{\infty} nh\nu e^{-nh\nu/kT}}{\sum_{n=0}^{\infty} e^{-nh\nu/kT}} \tag{3}$$

Substituting $x = e^{-h\nu/kT}$, we can write this as

$$\bar{\mathcal{E}} = h\nu \frac{\sum_{n=0}^{\infty} nx^n}{\sum_{n=0}^{\infty} x^n} \tag{4}$$

Now, we use the following formula from geometric series

$$\sum_{n=0}^{\infty} x^n \to 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \tag{5}$$

First differentiating both sides with respect to x and then multiplying both sides with x, we get

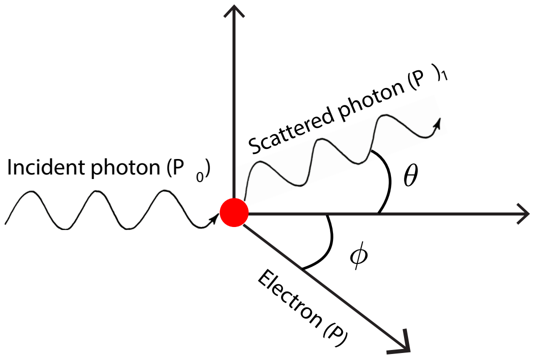
$$\begin{aligned} 1 + 2x + 3x^2 + 4x^3 + \cdots &= \frac{1}{(1-x)^2} \\ x + 2x^2 + 3x^3 + 4x^4 + \cdots &= \frac{x}{(1-x)^2} \\ \text{therefore} \qquad \sum_{n=0}^{\infty} nx^n &= \frac{x}{(1-x)^2} \end{aligned}$$

Substituting these formulas in the expression for average energy above, we get

$$\bar{\mathcal{E}} = h\nu \frac{\sum_{n=0}^{\infty} nx^n}{\sum_{n=0}^{\infty} x^n} = h\nu \frac{x/(1-x)^2}{1/(1-x)} = h\nu \frac{x}{1-x} = h\nu \frac{1}{(1/x) - 1} = \frac{h\nu}{e^{h\nu/kT} - 1} \tag{6}$$

Problem 3: For a photon of wavelength λ_0, scattering off of an electron of rest mass m_0, find the relationship between the photon scattering angle θ and the electron scattering angle ϕ. **(25 marks)**

Answer:



The conservation of momentum requires that

$$P_0 = P_1 cos\theta + P cos\phi \quad (7)$$

$$P sin\theta = P sin\phi \quad (8)$$

From equation (7) and (8) we get

$$\frac{P_0 - P_1 cos\theta}{cos\phi} = \frac{P_1 sin\theta}{sin\phi} \quad (9)$$

$$\frac{h}{\lambda_0} - \frac{h}{\lambda_1} cos\theta = \frac{h}{\lambda_1}\frac{sin\theta}{tan\phi} \quad (10)$$

$$\frac{1}{\lambda_0} = \frac{1}{\lambda_1}(cos\theta + \frac{sin\theta}{tan\phi}) \quad (11)$$

$$\frac{\lambda_1}{\lambda_0} = cos\theta + \frac{sin\theta}{tan\phi} \quad (12)$$

From the homework 2.1 (a) we already have that

$$\lambda_1 - \lambda_0 = \frac{h}{m_0 c}(1 - cos\theta) \quad (13)$$

$$\frac{\lambda_1}{\lambda_0} = 1 + \frac{h}{m_0 c \lambda_0}(1 - cos\theta) \quad (14)$$

From equation (12) and (14) we can write

$$cos\theta + \frac{sin\theta}{tan\phi} = 1 + \frac{h}{\lambda_0 m_0 c}(1 - cos\theta) \quad (15)$$

$$\frac{sin\theta}{tan\phi} = (1 - cos\theta)[1 + \frac{h}{\lambda_0 m_0 c}] \quad (16)$$

$$\frac{sin(\frac{\theta}{2})cos(\frac{\theta}{2})}{tan\phi} = 2sin^2(\frac{\theta}{2})[1 + \frac{h}{\lambda_0 m_0 c}] \quad (17)$$

$$cot(\frac{\theta}{2}) = tan\phi[1 + \frac{h}{\lambda_0 m_0 c}] \quad (18)$$

Problem 4: For an electron microscope to have a resolution of 0.1 Å, what is the minimum kinetic energy that the electron needs to have? ($m_0c^2 = 0.511$ MeV for an electron.) **(20 marks)**

Answer:
Here we have $\lambda = 0.1$Å. For an electron, the total relativistic energy

$$E = \sqrt{p^2c^2 + m_0^2c^4} Joule$$
$$m_0c^2 = 0.511 \times 10^6 eV$$

Therefore,

$$E = \sqrt{\frac{hc}{\lambda}(\frac{1}{1.602 \times 10^{-19}})^2 + (0.511 \times 10^6)^2} = 0.5258 \times 10^6 eV$$

So, the required energy is

$$K = E - m_0c^2 = (0.5258 - 0.511) \times 10^6 = 14.8 KeV$$