Department of Physics, IIT Kanpur, Semester II, 2016-17

PSO201A: Quantum Physics **Quiz # 2 Solutions** **Max Marks: 100**

Name: .... **Roll No.:** .... **Section**..........

Problem 1: The position-space wave function for a particle in the lowest-energy stationary state of a harmonic potential is given by $\psi_0(x) = (\frac{1}{2\pi\sigma_x^2})^{1/4}\exp(-\frac{x^2}{4\sigma_x^2})$, where $\sigma_x = \sqrt{\hbar/(2m\omega)}$.

(a) Find the momentum-space wave function for the particle. (Hint: you could make use of the standard integral: $\int_{-\infty}^{\infty} e^{-\alpha x^2+\beta x}dx = \exp\left(\frac{\beta^2}{4\alpha}\right)\left(\frac{\pi}{\alpha}\right)^{1/2}$.) **(15 marks)**

(b) Taking the standard deviation of a probability-density curve to be the uncertainty, calculate the position and momentum uncertainly product for the above wave-function. **(15 marks)**

Solutions:

(a) The momentum-space wave function can be obtained from the position-space wave-function as

$$\begin{aligned}
\psi_0(p) &= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi_0(x)e^{-ipx/\hbar}dx \\
&= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\left(\frac{1}{2\pi\sigma_x^2}\right)^{1/4}\exp\left(-\frac{x^2}{4\sigma_x^2}\right)e^{-ipx/\hbar}dx \\
&= \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{1}{2\pi\sigma_x^2}\right)^{1/4}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{4\sigma_x^2}x^2 - \frac{ip}{\hbar}x\right)dx \\
&= \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{1}{2\pi\sigma_x^2}\right)^{1/4}\exp\left[\frac{\left(\frac{-ip}{\hbar}\right)^2}{4\frac{1}{4\sigma_x^2}}\right]\left(\frac{\pi}{\frac{1}{4\sigma_x^2}}\right)^{1/2}
\end{aligned}$$

The above equation can be simplified to

$$\psi_0(p) = \left(\frac{1}{2\pi\sigma_p^2}\right)^{1/4}\exp\left(-\frac{p^2}{4\sigma_p^2}\right),$$

where $\sigma_p = \sqrt{\frac{\hbar m\omega}{2}}$.

(b) The position probability density is given by $P(x) = \psi_0^*(x)\psi_0(x) = (\frac{1}{2\pi\sigma_x^2})^{1/2}\exp(-\frac{x^2}{2\sigma_x^2})$. The standard deviation of this probability density is σ_x. Similarly, the momentum probability density $P(p)$ is given by $P(p) = \psi_0^*(p)\psi_0(p)$. Using the derived wave function above, we get $P(p) = \left(\frac{1}{2\pi\sigma_p^2}\right)^{1/2}\exp\left(-\frac{p^2}{2\sigma_p^2}\right)$. The standard deviation of the momentum-probability density is therefore σ_p. Taking the standard deviations to be the corresponding uncertainties, we get the position momentum uncertainty product to be $\Delta x\Delta p = \sigma_x\sigma_p = \sqrt{\hbar/(2m\omega)}\sqrt{\frac{\hbar m\omega}{2}} = \hbar/2$.

Problem 2: Let $\hat{H} = \begin{pmatrix} \hbar\omega & 0 \\ 0 & 2\hbar\omega \end{pmatrix}$ be the Hamiltonian and $\hat{A} = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}$ be an observable of a two-level system. Let $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ be the state of this system at $t = 0$.

(a) Calculate the normalized eigenvectors and the eigenvalues of the Hamiltonian. **(10 marks)**

(b) Calculate the expectation value of energy and of the observable $\hat{A}$ at $t = 0$. **(10 marks)**

(c) Find the 2×2 matrix representation for the propagator $\hat{U}(t, 0)$. **(10 marks)**

(d) Calculate the expectation value of energy and of the observable $\hat{A}$ at time t. **(10 marks)**

Solutions

(a) Since $\hat{H}$ is a diagonal matrix, its diagonal entries are the eigenvalues. So, we have $E_1 = \hbar\omega$ and $E_2 = 2\hbar\omega$ as the two eigenvalues and the corresponding normalized eigenvectors are $|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(b) The required expectation values are

$$\langle E(0)\rangle = \langle\psi(0)|\hat{H}|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} \hbar\omega & 0 \\ 0 & 2\hbar\omega \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{3}{2}\hbar\omega$$

$$\langle A(0)\rangle = \langle\psi(0)|\hat{A}|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mu$$

(c) The propagator $\hat{U}(t, 0)$ can be written as

$$\hat{U}(t,0) = |\psi_1\rangle\langle\psi_1|e^{-i\omega t} + |\psi_2\rangle\langle\psi_2|e^{-2i\omega t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix}e^{-i\omega t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}e^{-2i\omega t} = \begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{-2i\omega t} \end{pmatrix}$$

(d) The state $|\psi(t)\rangle$ at time t can be calculated as

$$|\psi(t)\rangle = \hat{U}(t,0)|\psi(0)\rangle = \begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{-2i\omega t} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega t} \\ e^{-2i\omega t} \end{pmatrix}$$

The required expectation values are

$$\langle E(t)\rangle = \langle\psi(t)|\hat{H}|\psi(t)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\omega t} & e^{2i\omega t} \end{pmatrix}\begin{pmatrix} \hbar\omega & 0 \\ 0 & 2\hbar\omega \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega t} \\ e^{-2i\omega t} \end{pmatrix} = \frac{3}{2}\hbar\omega$$

$$\langle A(t)\rangle = \langle\psi(t)|\hat{A}|\psi(t)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\omega t} & e^{2i\omega t} \end{pmatrix}\begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega t} \\ e^{-2i\omega t} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\omega t} & e^{2i\omega t} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} \mu e^{-2i\omega t} \\ \mu e^{-i\omega t} \end{pmatrix}$$

$$= \frac{\mu}{2}(e^{-i\omega t} + e^{i\omega t}) = \mu\cos(\omega t)$$

Problem 3: For the potential potential:

$$V(x) = -V_0 \qquad \text{for} \quad -a < x < a$$
$$= 0 \qquad \text{for} \quad |x| > a$$

(a) Work out the transcendental equation for the allowed energies of the odd bound states. **(20 marks)**

(b) What is the minimum number of bound states that this potential can have? **(10 marks)**

Solutions:

(a) See Homework Solution 8.2 (a).

(b) See Homework Solution 8.2 (c). The minimum number of odd bound states is zero. The minimum number of even bound state for this potential is one. So, the total minimum number of bound state for this potential is one.