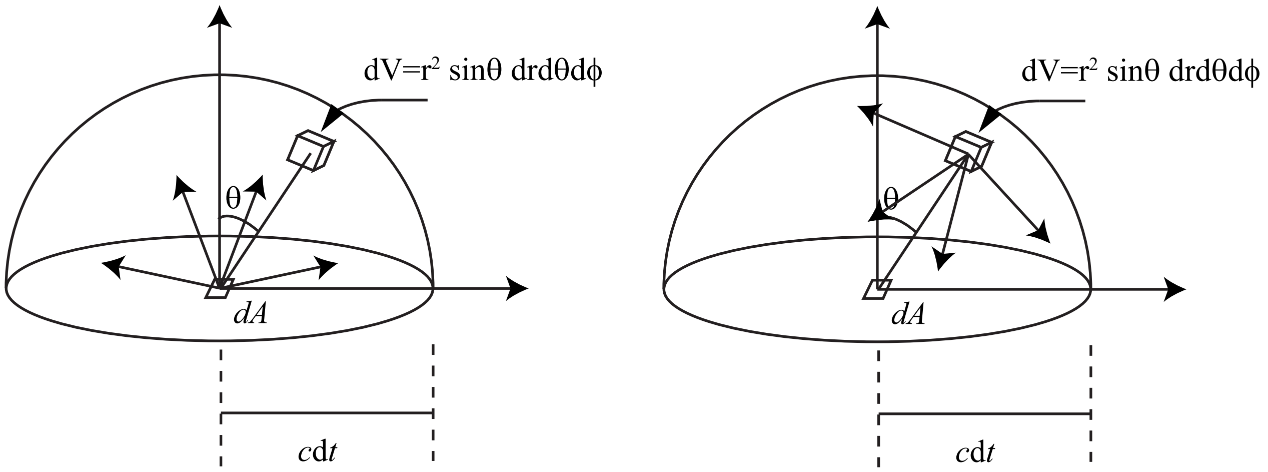
Department of Physics
IIT Kanpur, Semester II, 2016-17

PSO201A: Quantum Physics **Solution # 1**

Solution 1.1: Radiance and Energy density
In thermal equilibrium the total energy radiated out by an area element dA of a blackbody equals the energy incident on the element from the cavity (as illustrated in the figure below).



The energy E radiated out by an area element dA in time dt is

$$E = R(\nu)d\nu dA dt \tag{1}$$

This area element will receive energy from the hemispherical volume of radius cdt that is surrounding the area element. The radiation coming in from outside this region will not be able to reach the area element within dt. As shown in the figure, let's consider a volume element $dV = r^2 \sin\theta dr d\theta d\phi$ within the hemishpere. The total energy within this volume element is $\rho(\nu)d\nu dV = \rho(\nu)d\nu r^2 \sin\theta dr d\theta d\phi$. The energy dE falling from this volume element onto the area element dA which is at distance r and angle θ is $dE = \rho(\nu)d\nu r^2 \sin\theta dr d\theta d\phi \times \dfrac{dA}{4\pi r^2}\cos\theta$. Therefore the total energy falling onto the area element dA is

$$E = \rho(\nu)d\nu \frac{dA}{4\pi}\int_0^{cdt} dr \int_0^{\pi/2} \sin\theta\cos\theta d\theta \int_0^{2\pi} d\phi = E = \rho(\nu)d\nu dA dt\frac{c}{4} \tag{2}$$

Comparing the above two equations, we get the desired result, that is, $R(\nu) = \dfrac{c}{4}\rho(\nu)$.

Solution 1.2: Blackbody Radiation formula

(a) The classical expression for average energy is given by

$$\bar{\mathcal{E}} = \frac{\int_0^\infty \mathcal{E}P(\mathcal{E})d\mathcal{E}}{\int_0^\infty P(\mathcal{E})d\mathcal{E}} \tag{3}$$

Using the Boltzmann distribution $P(\mathcal{E}) = \dfrac{1}{kT}e^{-\mathcal{E}/kT}$ and substituting $x = \dfrac{\mathcal{E}}{kT}$, we can write the above expression as

$$\bar{\mathcal{E}} = kT\frac{\int_0^\infty xe^{-x}dx}{\int_0^\infty e^{-x}dx} = kT\frac{[e^{-x}(1+x)]_0^\infty}{[e^{-x}]_0^\infty} = kT\frac{0-1}{0-1} = kT \tag{4}$$

We find that the average energy $\bar{\mathcal{E}} = kT$ depends only on the temperature and does not depend on the frequency of the standing wave. This is also the statement of the law of equipartition of energy in classical theory. The energy $\rho(\nu)d\nu$ of standing waves inside a cavity can now be written as

$$\rho(\nu)d\nu = \frac{8\pi\nu^2 kT}{c^3}d\nu. \tag{5}$$

(b) Planck's hypothesis is that the standing wave inside a cavity can have energy only in the multiples of $h\nu$. With this form for the energy, the average energy of the standing waves becomes

$$\bar{\mathcal{E}} = \frac{\sum_{n=0}^{\infty} \mathcal{E}P(\mathcal{E})}{\sum_{n=0}^{\infty} P(\mathcal{E})} = \frac{\sum_{n=0}^{\infty} nh\nu e^{-nh\nu/kT}}{\sum_{n=0}^{\infty} e^{-nh\nu/kT}} \tag{6}$$

Substituting $x = e^{-h\nu/kT}$, we can write this as

$$\bar{\mathcal{E}} = h\nu \frac{\sum_{n=0}^{\infty} nx^n}{\sum_{n=0}^{\infty} x^n} \tag{7}$$

Now, we use the following formula from geometric series

$$\sum_{n=0}^{\infty} x^n \to 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \tag{8}$$

First differentiating both sides with respect to x and then multiplying both sides with x, we get

$$\begin{aligned} 1 + 2x + 3x^2 + 4x^3 + \cdots &= \frac{1}{(1-x)^2} \\ x + 2x^2 + 3x^3 + 4x^4 + \cdots &= \frac{x}{(1-x)^2} \\ \text{therefore} \qquad \sum_{n=0}^{\infty} nx^n &= \frac{x}{(1-x)^2} \end{aligned}$$

Substituting these formulas in the expression for average energy above, we get

$$\bar{\mathcal{E}} = h\nu \frac{\sum_{n=0}^{\infty} nx^n}{\sum_{n=0}^{\infty} x^n} = h\nu \frac{x/(1-x)^2}{1/(1-x)} = h\nu \frac{x}{1-x} = h\nu \frac{1}{(1/x) - 1} = \frac{h\nu}{e^{h\nu/kT} - 1} \tag{9}$$

The energy $\rho(\nu)d\nu$ of standing waves inside the cavity can now be written as

$$\rho(\nu)d\nu = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{e^{h\nu/kT} - 1} d\nu \tag{10}$$

Solution 1.3: Stefan's law

(a) Using the relation derived in Homework 1.1 and the Planck's radiation formula, we get

$$R(\nu)d\nu = \frac{c}{4}\rho(\nu)d\nu = \frac{2\pi h\nu^3}{c^2(e^{h\nu/kT} - 1)}d\nu. \tag{11}$$

So, the total radiance R_T is given by

$$R_T = \int_0^{\infty} R(\nu)d\nu = \frac{2\pi h}{c^2} \int_0^{\infty} \frac{\nu^3}{e^{h\nu/kT} - 1} d\nu \tag{12}$$

Using $x = \dfrac{h\nu}{kT}$ and $dx = \dfrac{hd\nu}{kT}$, we get

$$R_T = \frac{2\pi h}{c^2}\left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1} \tag{13}$$

Using the standard integral $\int_0^\infty \dfrac{x^3 dx}{e^x-1} = \dfrac{\pi^4}{15}$, we obtain

$$R_T = \frac{2\pi^5 k^4}{15c^2h^3}T^4 = \sigma T^4 \tag{14}$$

(b) Using $k = 1.38 \times 10^{-23}$ J/K, $c = 3 \times 10^8$ m/s, $h = 6.626 \times 10^{-34}$ J-s, we get $\sigma = 5.64 \times 10^{-8}$ W/m^2-K^4.

Solution 1.4: Wien's Displacement law

(a) First of all we need to express Planck's formula in terms of wavelength. We start from the equality $\rho(\lambda)d\lambda = -\rho(\nu)d\nu$. The minus sign indicates that $d\nu$ and $d\lambda$ have opposite signs (when one increases the other one decreases). Next, using $\nu = c/\lambda$, we get $d\nu = -\dfrac{c}{\lambda^2}d\lambda$. We can therefore write,

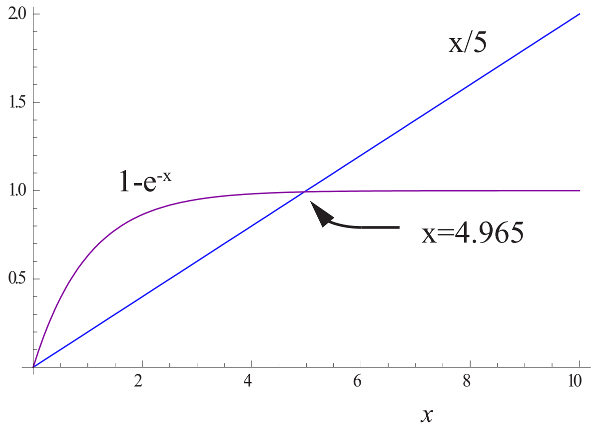
$$\rho(\lambda)d\lambda = -\rho(\nu)d\nu = -\frac{8\pi h(c/\lambda)^3}{c^3}\frac{1}{e^{hc/\lambda kT}-1}\left(\frac{-c}{\lambda^2}\right)d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}d\lambda$$

We are interested in finding the wavelength at which $\rho(\lambda)$ is maximum. Thus we need to solve the equation $d\rho(\lambda)/d\lambda = 0$, doing which we get

$$8\pi ch\left[\frac{-5}{\lambda^6}\frac{1}{e^{hc/\lambda kT}-1} + \frac{1}{\lambda^5}\frac{-e^{hc/\lambda kT}}{(e^{hc/\lambda kT}-1)^2}\frac{-hc}{\lambda^2 kT}\right] = 0 \tag{15}$$

Substituting $x = \dfrac{hc}{\lambda kT}$, we can write the above equation as

$$e^{-x} + \frac{x}{5} - 1 = 0 \tag{16}$$



This equation can be graphically solved to give $x = 4.965$ as the solution (see the figure). Using this solution we get $\dfrac{hc}{\lambda_{max}kT} = 4.965$. Therefore, we get

$$\lambda_{max}T = \frac{hc}{4.965 \times k} = \frac{6.626 \times 10^{-34} 3 \times 10^{-8}}{4.965 \times 1.381 \times 10^{-23}} = 2.898 \times 10^{-3}\text{m} - \text{K}. \tag{17}$$

This is Wien's displacement law.

(b) For an object that is at T =3000 K, the $\lambda_{\max}$ is

$$\lambda_{\max} = 2.898 \times 10^{-3}/3000 = 0.966 \times 10^{-6}\text{m} = 966\text{nm} \tag{18}$$

Since the $\lambda_{\max}$ is at 966 nm, the object would appear reddish.

Solution 1.5: Photoelectric effect

(a) Suppose the eye can detect a minimum of N photons. Then $N\dfrac{hc}{\lambda} = 1 \times 10^{-18}$ J. Therefore,

$$N = \frac{\lambda}{hc} 1 \times 10^{-18} = \frac{1 \times 10^{-18} \times 600 \times 10^{-9}}{6.626 \times 10^{-34} \times 3 \times 10^{8}} \approx 3 \ \text{ photons} \tag{19}$$

It is clear that the eyes are excellent detectors. They have the detection sensitivity of up to a few photons.

(b) If a photon is absorbed by an electron, momentum conservation requires that the electron now moves along the initial photon direction and so the momentum conservation requires

$$p_{ph} = \frac{h}{\lambda} = p_e \tag{20}$$

where p_{ph} and p_e are the momenta of the photon and the electron, respectively.

The energy conservation requires that

$$\frac{hc}{\lambda} + m_e c^2 = \sqrt{p_e^2 c^2 + m_e^2 c^4} \tag{21}$$

In order for both these conversation equations to be satisfied, we have to have (substituting for p_e in the above equation)

$$\begin{aligned} \left(\frac{hc}{\lambda} + m_e c^2\right)^2 &= \frac{h^2 c^2}{\lambda^2} + m_e^2 c^4 \\ \text{or,} \quad \frac{2hc}{\lambda} m_e c^2 &= 0 \end{aligned} \tag{22}$$

This is an impossible condition to satisfy since the rest mass of an electron is non-zero. Thus we conclude that it is impossible for a photon to give up all its energy and momentum to a free electron.

(c) In case of photoelectric effect with bound electrons, the atoms, which are much heavier than electrons also come into picture and play the most significant role. Since atoms are heavier they can absorb a large amount of extra momentum without taking away too much of the energy. This way a photon can be absorbed and an electron emitted while satisfying both the momentum and energy conservations.

(d) Since the work function of Sodium is 2.3 eV. The cut-off wavelength is

$$\lambda_0 = \frac{hc}{2.3\text{eV}} = \frac{6.626 \times 10^{-34} \times 3 \times 10^{8}}{2.3 \times 1.6 \times 10^{-19}} \approx 541 \text{ nm} \tag{23}$$

So, in order to eject an electron from Sodium, the wavelength of the incoming light must be smaller than 541 nm. A one Watt, HeNe laser at 633 nm is very intense and has a lot of photons but it does not have photons that are energetic enough to eject out an electron from Sodium. Therefore, there is no probability of electron ejection from sodium. On the other hand, although the torch-light of a cell-phone is a much less intense source than a laser, it does have photons at wavelengths lower than 541 nm. And therefore, there is a finite probability that it will be able to eject electron from sodium.