Department of Physics
IIT Kanpur, Semester II, 2016-17

PSO201A: Quantum Physics **Solution # 2**

Solution 2.1: Radiation manifesting as particles

(a) Follow the derivation given in Section 2.4 of Eisberg and Resnick.

(b) Follow the derivation given in Example 2.4 (b) of Eisberg and Resnick.

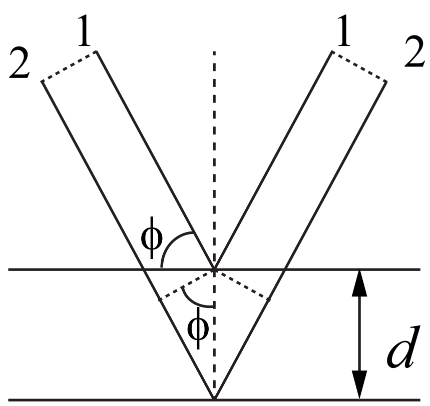
(c) The Compton shift $\Delta\lambda$ at $\theta = \pi/2$ is independent of the wavelength of the incident wave and is given by $\Delta\lambda = \lambda_c = 0.0243$ Å.

(d) For the observation of the Compton effect experimentally, the relevant quantity that we need to consider is $\dfrac{\Delta\lambda}{\lambda_0}$

$$\text{for } \gamma \text{ rays} : \lambda_0 = 1.88 \times 10^{-2}\text{Å}; \quad \text{and} \quad \frac{\Delta\lambda}{\lambda_0} = 1.29$$
$$\text{for X} - \text{rays} : \lambda_0 = 1.0\text{Å}; \quad \text{and} \quad \frac{\Delta\lambda}{\lambda_0} = 2.43 \times 10^{-2}$$
$$\text{for visible photons} : \lambda_0 = 5000\text{Å}; \quad \text{and} \quad \frac{\Delta\lambda}{\lambda_0} = 4.86 \times 10^{-6}$$

We notice that it will be much easier to see the effect with γ-rays since the two wavelengths λ_0 and λ_1 are well separated. With X-rays, it will be relatively more difficult since the two wavelengths λ_0 and λ_1 are close to within two percent of the wavelength. With visible light, it will be impossible to see the Compton effects since the two wavelengths λ_0 and λ_1 can almost not be separated out in measurements.

Solution 2.2: Electrons (material particle) manifesting as waves



(a) Refer to the figure. The extra length travelled by the second ray is $d\sin\phi + d\sin\phi = 2d\sin\phi$. So, for the two rays to interfere constructively, this path difference should be an integral multiples of λ, the wavelength of the incoming wave (or electrons), that is, $2d\sin\theta = n\lambda$. This is Bragg's reflection condition.

(b) For the given kinetic energy, the de-Broglie wavelength can be calculated to be

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} = \frac{6.6 \times 10^{-34}}{2 \times 9.1 \times 10^{-31} \times 60 \times 1.6 \times 10^{-19}} = 1.57\text{Å}$$

(c) Since $n = 1$, we have $2d\sin\phi = \lambda$. This gives $\sin\phi = \dfrac{\lambda}{2d} = \dfrac{1.57}{2 \times 0.91} = 0.862$, that is, $\phi = 59.5^0$.

(d) Here we have $n = 3,\ \phi = 59.5^0, d = 0.91$ Å. Therefore the required wavelength is given by $\lambda = \frac{2}{3} d \sin\phi = 0.523$ Å. The required energy of the photons is therefore

$$E = \frac{p^2}{2m} = \left(\frac{h}{\lambda}\right)^2 \frac{1}{2m} = \frac{(6.6 \times 10^{-34})^2}{(0.523 \times 10^{-10})^2 2 \times 9.1 \times 10^{-31}} = 550 \text{ eV}.$$

Solution 2.3: Young's double-slit interference with particles

(a) The form of the wave amplitude is given by $E(\vec{\boldsymbol{r}}, t) = Ae^{i(kz-\omega t)}$. So, the incident wave amplitude at the double-slit plane ($z = 0$) is of the form $E(\vec{\boldsymbol{r}}, t) = Ae^{-i\omega t}$. We are interested in finding the intensity at the observation point $(\vec{\boldsymbol{r}_0}, t)$. The total field $E(\vec{\boldsymbol{r}_0}, t)$ at the observation point is equal to the sum of the fields coming from the two slits. This can be written as

$$E(\vec{\boldsymbol{r}_0}, t) = C_0 E(\vec{\boldsymbol{r}_1}, t - t_1) + C_0 E(\vec{\boldsymbol{r}_2}, t - t_2).$$

Here C_0 is a constant that appears because only a fraction of the field present at the slits will reach the observation point. Using the form of the incident field at the double-slit plane, we write the above equation as

$$E(\vec{\boldsymbol{r}_0}, t) = C_0 A e^{-i\omega(t-t_1)} + C_0 A e^{-i\omega(t-t_2)}$$

Here $t_1 = \frac{R_1}{v}$ and $t_2 = \frac{R_2}{v}$, where v is the speed of the incoming waves. Also substituting $\omega = \frac{2\pi v}{\lambda}$, the above expression can be written as

$$\begin{aligned} E(\vec{\boldsymbol{r}_0}, t) &= C_0 A \exp\left[-i\left[\frac{2\pi v}{\lambda}\left(t - \frac{R_1}{v}\right)\right]\right] + C_0 A \exp\left[-i\left[\frac{2\pi v}{\lambda}\left(t - \frac{R_2}{v}\right)\right]\right] \\ &= C_0 A \exp\left[-ik\left(vt - R_1\right)\right] + C_0 A \exp\left[-ik\left(vt - R_2\right)\right] \end{aligned}$$

The expression for the intensity at the screen now takes the following form

$$I(\vec{\boldsymbol{r}_0}, t) = |E(\vec{\boldsymbol{r}_0}, t)|^2 = |C_0 A|^2 \left[1 + \cos k(R_2 - R_1)\right]$$

Using x, d and R, as shown in the figure, $R_2 - R_1$ can be written as

$$\begin{aligned} R_2 - R_1 &= \sqrt{R^2 + \left(x + \frac{d}{2}\right)^2} - \sqrt{R^2 + \left(x - \frac{d}{2}\right)^2} \\ &= R\left[1 + \frac{(x + \frac{d}{2})^2}{R^2}\right]^{1/2} - R\left[1 + \frac{(x - \frac{d}{2})^2}{R^2}\right]^{1/2} \\ &\approx R\left[1 + \frac{(x + \frac{d}{2})^2}{2R^2}\right] - R\left[1 + \frac{(x - \frac{d}{2})^2}{2R^2}\right] \\ &= R\left[\frac{(x + \frac{d}{2})^2}{2R^2} - \frac{(x - \frac{d}{2})^2}{2R^2}\right] = R\left[\frac{2xd}{2R^2}\right] = \frac{xd}{R} \end{aligned}$$

The intensity expression above can now be written as

$$I(\vec{\boldsymbol{r}_0}, t) = |C_0 A|^2 \left[1 + \cos\left(\frac{kxd}{R}\right)\right]$$

The fringe period is therefore equal to $\frac{2\pi R}{kd} = \frac{\lambda R}{d}$.

(b) For $R = 1$ m, $d = 1$ mm, and $\lambda = 5000$ Å. The fringe period is

$$\frac{\lambda R}{d} = \frac{5000 \times 10^{-10} \times 1}{1 \times 10^{-3}} = 5 \times 10^{-4} \text{ m} = 0.5 \text{ mm}.$$

(c) For a bullet of mass $m = 60$ g moving with speed of $v = 200$ m/s, the de-Broglie wavelength is $\lambda = \frac{h}{p} = \frac{6.6 \times 10^{-34}}{0.06 \times 200} = 5.5 \times 10^{-35}$ m. For $R = 1$ m, $d = 1$ mm, the fringe period is

$$\frac{\lambda R}{d} = \frac{5.5 \times 10^{-35} \times 1}{1 \times 10^{-3}} = 5.5 \times 10^{-32} \text{ m}$$

(d) Since the fringe period is very very small, it is impossible to observe the fringes. To be able to see such fringes, we need bullet-detectors with spatial resolution smaller than 10^{-32} m. However, any realistic bullet-detector has the spatial resolution in centimeters (size of the bullet). So, what is detected by these bullet-detectors is intensity averaged over many fringe periods and therefore a bullet detector never sees the interference pattern.

Solution 2.4: Miscellaneous Conceptual Questions

(a) Increasing the intensity certainly increases the energy per unit area per unit time but when we have photons increasing the intensity only means increasing the number of photons falling per unit area per unit time. However, the energy of individual photons depends only on frequency and doesn't depend on the intensity. In the photoelectric effect, the kinetic energy of the ejected electron depends on the energy of the individual incoming photons and so it remains independent of the intensity.

(b) In a Young's double-slit experiment with particles (photons, electorns, etc.), individual particles go one-at-a-time through one of the two slits and over a period of time one observes intensity fringes on a screen placed at some distance from the double-slit plane. Since the individual particles pass through one-at-a-time, one can ask as to which slit the individual particles pass through. One can get the answer to this question, for example, by putting some kind of particle detector on the slits itself. However, if one gains the information as to which slit the particles go through, one looses the interference on the screen. In fact, the degree to which one knows the which-slit information (particle behaviour), one looses the interference visibility to precisely the same degree (wave behaviour). This is Bohr's complementarity principle.

(c) The main difference between the wave-function $\psi(x,t)$ representing a quantum particle and the function $E(x,t)$ representing the wave-amplitude of a classical wave is that division of wave-amplitude (as in interference experiments) also implies division of energy but the division of wave-function (again as in interference experiments) does not imply the division of energy.

(d) A photon interferes with itself.